Let me count the ways ... or complex analysis meets complexity analysis

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Don't get deranged!

- The derangement problem:
  - A *derangement* is permutation π of an n-element set S_n which is *1-cyclefree*, i.e., if it has no fixed points: there is no $s \in S_n$ s.th. $\pi(s) = s$
  - For $n = 4$, $S_n = 1, 2, 3, 4$ the following permutations (out of 24) are derangements:

$$2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321$$

  - Q: What is the probability that a randomly chosen permutation of S_n is a derangement? How does it behave as n grows?
  - A: For large $\sharp S$ this probability approaches $1/e = 0.367879...$

This one is a little trickier..

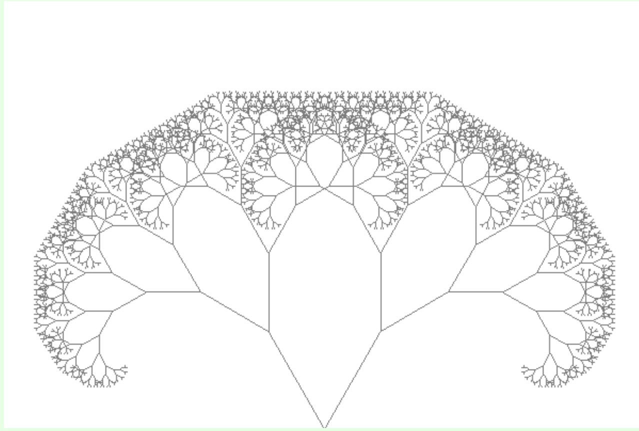
- Another derangement problem
  - A permutation π of an n-element set S is a *1-2-cyclefree* if it has no fixed points and no cycles of length 2 (i.e. for all $s \in S$: $\pi(s) \neq s$ and $\pi^2(s) \neq s$)
  - For $n = 4$, $S_n = \{1, 2, 3, 4\}$ the following permutations (out of 24) are 1-2-cyclefree:

$$2341, 2413, 3142, 3421, 4123, 4312$$

  - Q: What is the probability that a randomly chosen permutation of S is 1-2-cyclefree? How does it behave as n grows?
  - A: For large $\sharp S$ this probability approaches $e^{-3/2} = 0.22313...$

What is the shape of a typical tree?

- What does a typical (=random) large binary tree look like?
- Like this?



Probably not ....

What is the shape of a typical tree?

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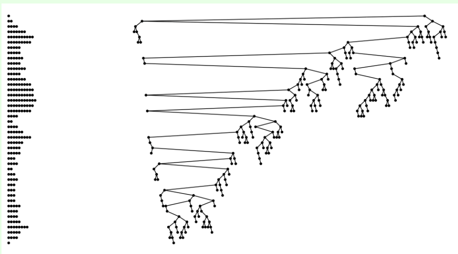
- Like this?



- Or like this?

What is the shape of a typical tree?

- So what is typically the height, width, shape, ... of a binary tree?
- If you don't have an answer, you might try experimentally by sampling
- But how do you sample from binary trees?

Counting rabbits à la Fibonacci

- Consider the sequence of Fibonacci numbers $(f_n)_{\geq 0}$ defined by

$$f_{n+1} = f_n + f_{n-1} \quad (n \geq 1) \quad f_0 = 0, f_1 = 1$$

- First values:

n	0	1	2	3	4	5	6	7	8	9	10
f_n	0	1	1	2	3	5	8	13	21	34	55

- $f_{100} = 354224848179261915075$
- Q: How fast does this sequence grow?
- A: Easy because the recurrence is linear with constant coefficients:

$$f_n = \frac{\phi^n - \hat{\phi}^n}{\sqrt{5}} \quad \text{where} \quad \begin{array}{l} \phi = \frac{1+\sqrt{5}}{2} \approx 1.61803 \\ \hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -0.618034 \end{array}$$

The analytic picture

- Consider the power series (a.k.a. "generating function")

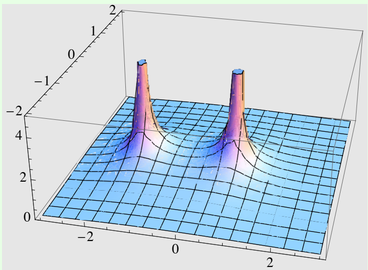
$$f(z) = \sum_{n \geq 0} f_n\, z^n = z + z^2 + 2z^3 + 3z^4 + 5z^5 + \cdots$$

- The recurrence is equivalent to the rational function

$$\begin{aligned} f(z) &= \frac{z}{1 - z - z^2} \\ &= \frac{z}{(1-\phi\, z)(1-\hat{\phi}\, z)} \\ &= \frac{1}{\sqrt{5}} \left(\frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z} \right) \end{aligned}$$

The analytic picture contd.

- Look at the plot of $|f(z)|$ for complex z



- There are two "singularities" where the denominator vanishes: $z = \phi^{-1} = 0.618034$ and $z = \hat{\phi}^{-1} = -1.61803$.
  ϕ^{-1} is the "dominant singularity" which determines the growth rate of $(f_n)_{n \geq 0}$.
- $\rho = \phi^{-1}$ is the radius of convergence of the series $f(z)$

Dyck-language the "typical" context-free language

- D = the language of properly nested parentheses pairs () alias 01
- (unambiguous) context-free grammar

$$\mathcal{D} : D \to \varepsilon \mid D\,0\,D\,1$$

- $D_n = \{w \in D\,;\, |w| = 2n\}$, $d_n = \sharp D_n$
- First sets

$$\begin{array}{l} D_0 = \{\varepsilon\} \quad D_1 = \{01\} \quad D_2 = \{0101, 0011\} \\ D_3 = \{010101, 010011, 00110, 001011, 000111\} \end{array}$$

- The derivation trees of $\mathcal{D}$ are precisely the binary trees
  words in D_n encode binary trees with n interior nodes and $n+1$ leaves

(Euler-Segner-) Catalan numbers (ctd.)

- cardinalities: $d_n = \sharp D_n$

n	0	1	2	3	4	5	6	7	8	9	10
d_n	1	1	2	5	14	42	132	429	1430	4862	16796

- for big n numbers d_n can be computed easily e.g.

$d_{100} =$
896519947090131496687170070074100632420837521538745909320

(Euler-Segner-) Catalan numbers (ctd.)

- The numbers d_n
  1. satisfy a nonlinear recurrence
  $$d_{n+1} = d_0\, d_n + d_1\, d_n + \cdots + d_n\, d_1 \tag{1}$$
  2. satisfy a *first-order* linear recurrence with *polynomial* coefficients
  $$(n+2)\, d_{n+1} = 2(2n+1)\, d_n \tag{2}$$
  3. have a “closed form”
  $$d_n = \frac{1}{n+1}\binom{2n}{n} \tag{3}$$
- (1) follows from the grammar (2) and (3) are obviously equivalent validity of (2) can be seen from looking at binary trees (3) has a neat combinatorial proof using cyclic shifts of balanced words

Asymptotics of the Catalan numbers

- From the “closed form” (2) the asymptotic behaviour of the sequence $(d_n)_{n\geq 0}$ can be obtained easily:
  - Use Stirling's formula
  $$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as} \quad n \to \infty$$
  to estimate the binomial coefficient
  - to obtain
  $$d_n \sim \frac{4^n}{\sqrt{2\pi}\; n^{3/2}} \quad \text{as} \quad n \to \infty$$

Asymptotics of the Catalan numbers

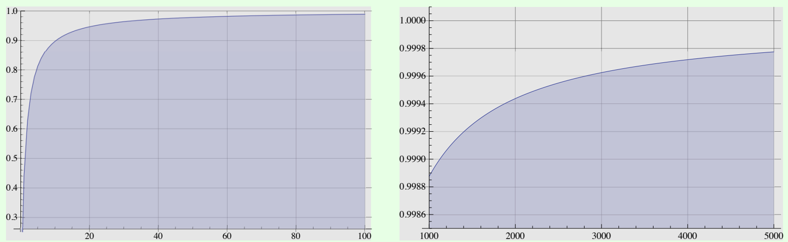


Figure: Relative approximation of Catalan numbers for n=0..100 and for n=1000..5000

The (Chomsky-) Schützenberger-Theorems

- $L \subset \Sigma^*$ a formal language
- $\ell_n = \sharp(L \cap \Sigma^n)$ number of words of length n in L
- $f_L(z) = \sum_{n \geq 0} \ell_n z^n$ the "generating function" of L
- (Chomsky-) Schützenberger-Theorems
  1. If L is regular (i.e. type-3) then $f_L(z)$ is a rational function i.e. there are polynomials $p(z), q(z)$ s.th.
  $$f_L(z) = \frac{p(z)}{q(z)}$$
  $\Rightarrow (\ell_n)_{n \geq 0}$ satisfies a linear recurrence with constant coefficients
  2. If L is unambiguously context-free (type-2 unambig) then $f_L(z)$ is an algebraic function i.e. there is a polynomial $P(z, y)$ such that
  $$P(z, f_L(z)) = 0$$
  $\Rightarrow (\ell_n)_{n \geq 0}$ satisfies a linear recurrence with polynomial coefficients

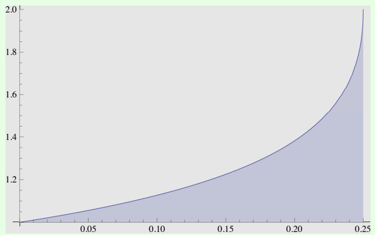
Our first example: $L = F$ (Fibonacci)

- A regular language for Fibonacci: $F = 0.(0 + 11)^* \subseteq \{01\}^*$
- $F_n = F \cap \{01\}^n$ with
$$F_{n+1} = F_n.0 + F_{n-1}.11 \quad F_0 = \emptyset, F_1 = \{0\}$$
- First sets:
$$F_0 = \emptyset$$
$$F_1 = \{0\} \quad F_2 = \{00\} \quad F_3 = \{000, 011\}$$
$$F_4 = \{0000, 0011, 0110\}$$
$$F_5 = \{00000, 00011, 00110, 01100, 01111\}$$
- The sequence $(f_n)_{n \geq 0} = (\sharp F_n)_{n \geq 0}$ satisfies a second-order linear recurrence with constant coefficients and the generating function
$$\sum_{n \geq 0} f_n \, z^n = \frac{z}{1 - z - z^2}$$
is rational

Our second example: $L = D$ (Dyck)

- From the basic recurrence
$$f_D(z) = 1 + z \, f_D(z)^2$$
and thus
$$f_D(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$
- Expanding the radical (Newton's binomial theorem) gives Catalan numbers again:
$$f_D(z) = \sum_{n \geq 0} d_n \, z^n = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} z^n$$

The function $f_D(z)$



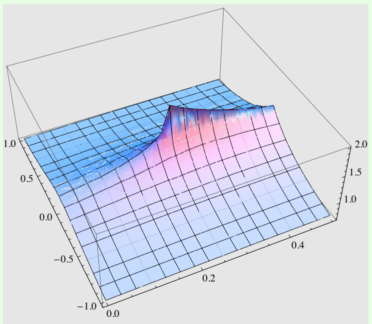


Figure: Graph of $f_D(z)$ for $0 \leq z \leq 1/4$ (left)
and of $|f_D(z)|$ for $0 \leq \Re(z) \leq 0.5$ and $-1 \leq \Im(z) \leq 1$ (right)

- $z = 1/4$ is an "algebraic singularity" of $f_D(z)$ indeed the only singularity of $f_D(z)$ and also the radius of convergence of the series

D is not rational!

- Generating function argument:
  - ▷ $f_D(z) = \dfrac{1-\sqrt{1-4z}}{2z}$ is not a rational function!
- Asymptotic argument:
  - ▷ Any sequence $(a_n)_{n\geq 0}$ that satisfies a linear recurrence with constant coefficients behaves asymptotically like
$$a_n \sim p(n)\,\lambda^n \quad \text{as} \quad n \to \infty$$
where $p(\,.\,)$ is a polynomial
  - ▷ We have seen
$$d_n \sim \frac{4^n}{\sqrt{2\pi}\, n^{3/2}} \quad \text{as} \quad n \to \infty$$

Why counting?

- In fields like
  - Probability (random generation)
  - Physics (statistical mechanics)
  - Chemistry (organic structures)
  - Algorithm analysis (average-case complexity)

  important problems can be reduced to counting
- Information about the quantitative behaviour of systems can be deduced from the asymptotic behaviour of "number sequences"
- Asymptotics
  - is usually easy if (exact) "closed" formulas are available — which is rarely the case
  - is (often) feasible if "nice" recurrences are available
  - is (often) feasible if the "generating functions" can be treated with methods of complex analysis (saddle point methods singularity analysis Mellin transforms...)

The scenario for average-case complexity

- $\mathcal{D}$: a family of objects (data)
- $\texttt{size} : \mathcal{D} \to \mathbb{N}$: a size-function of objects
  $\mathcal{D}_n$: objects of size n $d_n = \sharp\mathcal{D}_n$
- $\mathcal{A}$: an algorithm that operates on objects from $\mathcal{D}$
- $\texttt{cost}_{\mathcal{A}} : \mathcal{D} \to \mathbb{R}_{\geq 0}$: a cost-function for executing $\mathcal{A}$ on $\mathcal{D}$
  $c_n = \sum_{t\in\mathcal{D}_n} \texttt{cost}_{\mathcal{A}}(t)$: cumulated cost for $\mathcal{A}$ on $\mathcal{D}_n$
- average-case complexity of $\mathcal{A}$ on $\mathcal{D}_n$:
$$\frac{c_n}{d_n} = \frac{1}{d_n} \sum_{t\in\mathcal{D}_n} \texttt{cost}_{\mathcal{A}}(t)$$
- Goal: determine the asymptotic behaviour (growth rate) of the sequence $\left(\dfrac{c_n}{d_n}\right)_{n\geq 0}$ as $n \to \infty$

The problem of average-case complexity

- Associate with $\mathcal{D}$ and $\texttt{size}$ the generating function
$$d(z) = \sum_{n\geq 0} d_n\, z^n = \sum_{t\in\mathcal{D}} z^{\texttt{size}(t)}$$
  and with $\texttt{cost}$ the generating function
$$c(z) = \sum_{n\geq 0} c_n\, z^n = \sum_{t\in\mathcal{D}} \texttt{cost}_{\mathcal{A}}(t)\, z^{\texttt{size}(t)}$$
- Or (if $\texttt{cost}$ takes nonnegative integer values) take right away the bivariate generating function
$$w(u,z) = \sum_{t\in\mathcal{D}} u^{\texttt{cost}_{\mathcal{A}}(t)}\, z^{\texttt{size}(t)}$$
  and note that $d(z) = w(1,z)$, $\quad c(z) = \partial_u w(u,z)\big|_{u\leftarrow 1}$
- The problem is: functions $d(z), c(z), w(u,z)$ are almost never known explicitly! They are accessible only through functional equations they satisfy. How can one get asymptotics from that?

Analysis comes in ...

- Growth rates can be studied using generating functions. Why?
- Remember from your calculus class the Hadamard-criterion:
  - If the power series
    $$a(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$
    has radius of convergence ρ then
    $$\rho^{-1} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$
  - So one may expect as *exponential growth rate*:
    $$(|a_n|)_{n\geq 0} \quad \text{grows like} \quad \rho^{-n} \quad \text{as} \quad n \to \infty$$
    often written as: $a_n \asymp \rho^{-n}$
  - This is sufficient information in some cases but usually one has to take care of "subexponential factors" in order to get meaningful results

The guiding rules

- Given a sequence $(a_n)_{n\geq 0}$ one would like to estimate its growth rate as
  $$a_n = A^n \cdot \alpha(n)$$
  where $\alpha(n)$ grows sub-exponentially (or is even bounded)
- Basic insight:
  - The exponential growth rate of a sequence $(a_n)_{n\geq 0}$ depends on the location of the dominant singularity — which for us is the radius of convergence ρ of $a(z) = \sum_{n\geq 0} a_n z^n$ so that $A = \rho^{-1}$
  - The associate subexponential factor $\alpha(n)$ depends on the nature of ρ as a singularity: rational algebraic logarithmic...
    One has to look for the behaviour of $a(z)$ as z approaches ρ

More analysis ... things get really complex

- Cauchy's integral formula
  If $f(z)$ is an analytic function in some domain $D \subseteq \mathbb{C}$ with $0 \in D$ and if $f(z) = a_0 + a_1 z + a_2 z^2 \cdots$ is its power series expansion at $z = 0$ then
  $$a_n = [z^n]\, f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z^{n+1}} dz$$
  where γ is any (!) simple closed path around $z = 0$ in D
- Asymptotics of the Newton series coefficients for $\alpha \notin -\mathbb{N}$:
  $$[z^n]\,(1-z)^{-\alpha} = \binom{n+\alpha-1}{n} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}$$
  $$\sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left[1 + \frac{\alpha(\alpha-1)}{2n} + \frac{\alpha(\alpha-1)(\alpha-2)(3\alpha-1)}{24n^2} + \cdots\right]$$

The transfer principle (Flajolet-Odlyzko)

- The main idea is
  $$a(z) \sim_{z\to\rho} \sigma(z) \quad \Rightarrow \quad [z^n]\; a(z) \sim [z^n]\; \sigma(z)$$
  where $\sigma(z)$ is a function usually much simpler than $a(z)$
- This holds under certain (mild, for our applications) conditions with approximating functions (for $\rho = 1$) like
  $$\sigma(z) = \left(1 - \frac{z}{\rho}\right)^\alpha \log^\beta\left(1 - \frac{z}{\rho}\right)$$

Simple cases of transfer

- Let $f(z) = \sum_{n\geq 0} f_n z^n$ be a power series with radius of convergence $\rho = 1$ and $f(1) \neq 0$. Then

$$[z^n]\,\frac{f(z)}{1-z} \;\sim\; f(1)$$

$$[z^n]\,f(z)\,\sqrt{1-z} \;\sim\; -\frac{f(1)}{2\sqrt{\pi n^3}}$$

$$[z^n]\,f(z)\,\log\frac{1}{1-z} \;\sim\; \frac{f(1)}{n}$$

Motzkin trees

Consider the following variant of binary trees: unary-binary trees (a.k.a. Motzkin trees)

- $\mathcal{M}$: trees where each internal node has either one or two (ordered) successors
- Written as a context-free grammar

$$\mathcal{M} : M \to \varepsilon \mid \star \mid M\,0\,M\,1$$

- $M_n = M \cap \{0,1,\star\}$, $m_n = \sharp M_n$
- First sets

$$\begin{aligned} M_0 &= \{\varepsilon\} \\ M_1 &= \{\star\} \\ M_2 &= \{\star\star, 01\} \\ M_3 &= \{\star\star\star, \star01, 0\star1, 01\star\} \\ M_4 &= \{\star\star\star\star, \star\star01, \star0\star1, \star01\star, 0\star\star1, 0\star1\star, 01\star\star, 0101, 0011\} \end{aligned}$$

Motzkin trees

- First values

n	0	1	2	3	4	5	6	7	8	9	10
m_n	1	1	2	4	9	21	51	127	323	835	2188

- Values can be computed quite easily

$$m_{100} = 737415571391164350797051905752637361193303669$$

- One has

$$m_n = \sum_{j\geq 0} \frac{1}{j+1}\binom{2j}{j}\binom{n}{2j}$$

  but there is no neat "closed form" of m_n
- The numbers satisfy a recurrence

$$(n+1)\,m_{n+1} = (2n+3)\,m_n + 3n\,m_{n-1}, \quad m_0 = m_1 = 1$$

  but it seems difficult to obtain asymptotic growth information

Motzkin trees: look at the generating function

- The generating function

$$m(z) = \sum_{n\geq 0} m_n z^n = 1 + z + 2z^2 + 4z^3 + 9\,z^4 + \cdots$$

  satisfies (from the grammar)

$$m(z) = 1 + z\cdot(m(z) + m(z)^2)$$

- and hence

$$m(z) = \frac{1 - z - \sqrt{1-2z-3z^2}}{2z^2}$$

- because $1-2z-3z^2 = (1+z)(1-3z)$ the critical values ("singularities") of $m(z)$ are $z=-1$ and $z = 1/3$
- $\rho = 1/3$ is the "dominant singularity" (=radius of convergence) — expect $m_n \asymp 3^n$

Motzkin trees: singularities visualized

- Look at $m(z)$ in the vicinity of $\rho = 1/3$

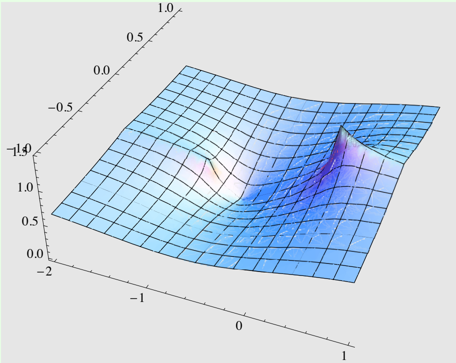


Figure: Plot of $|m(z)|$ for $-2 \leq \Re(z) \leq 1$ and $-1 \leq \Im(z) \leq 1$

Motzkin trees: look at the generating function (contd.)

- Expand $m(z)$ around $\rho = 1/3$:

$$m(z) \approx 3(1 - \sqrt{3}\sqrt{1-3z})$$

- This gives

$$m_n = \frac{3}{2}\sqrt{\frac{3}{\pi n^3}} \cdot 3^n \cdot \left(1 + \mathcal{O}(\frac{1}{n})\right)$$

- Looking closer one can get

$$m_n = \left(\frac{3}{2}\sqrt{\frac{3}{\pi n^3}} - \frac{117}{32}\sqrt{\frac{3}{\pi n^5}}\right) \cdot 3^n \cdot \left(1 + \mathcal{O}(\frac{1}{n^2})\right)$$

and more ...

Motzkin trees: look at the generating function (contd.)

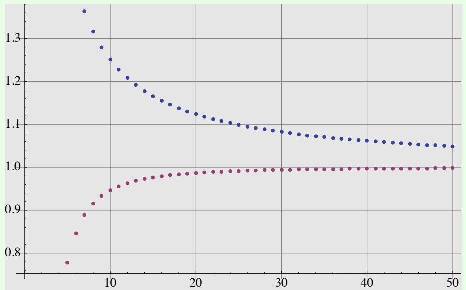
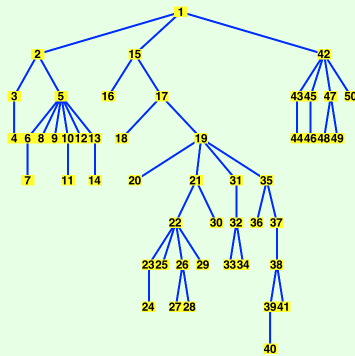


Figure: Two (relative) approximations of the $(m_n)_{1 \leq n \leq 50}$

Ordered trees

- Ordered trees (a.k.a. planted plane trees):
  - are rooted trees
  - with an arbitrary (finite) number of successors of each node
  - successors (subtrees) of a node are linearly ordered
- Example:



Ordered trees and their levels

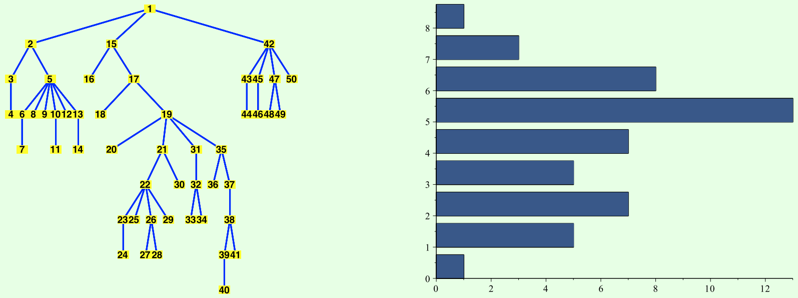


Figure: A random ordered tree with 50 nodes and its level distribution

Ordered trees and their levels

- `size` of an ordered tree = number of nodes
- Interesting parameters of ordered trees (of size n)
  - `height`
    – can be anything between 1 and $n-1$
  - `pathlength` ($\sim$ `average level` $\cdot$ `size`)
    – can be anything between between $n-1$ and $\binom{n}{2}$
  - level distribution
    – can be very different for different instances
- So, what is typical ?
  (i.e., averaging over all ordered trees of size n)
- Try to guess the answer from experiments!
  This need true random generation based on exact counting

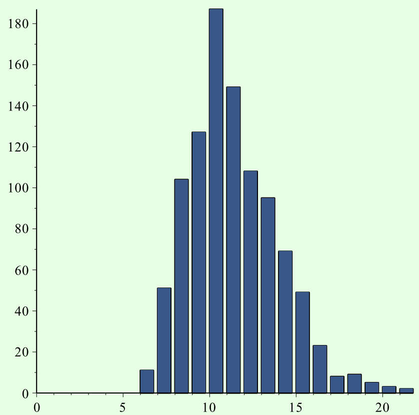


Figure: Height statistics for 1000 randomly generated trees of size 50

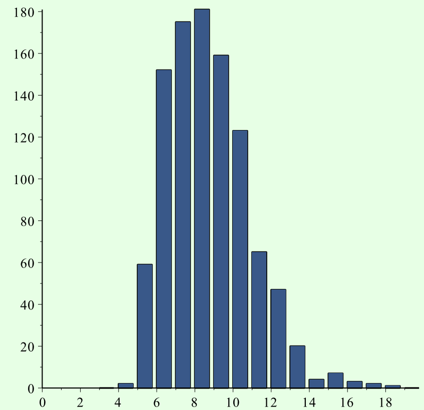


Figure: Average level statistics for 1000 randomly generated trees of size 100

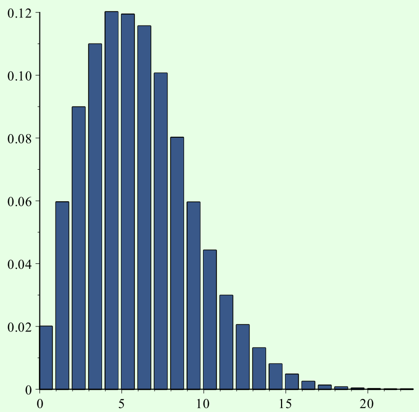


Figure: Average profile for 1000 randomly generated trees of size 50 (as a probability distribution)

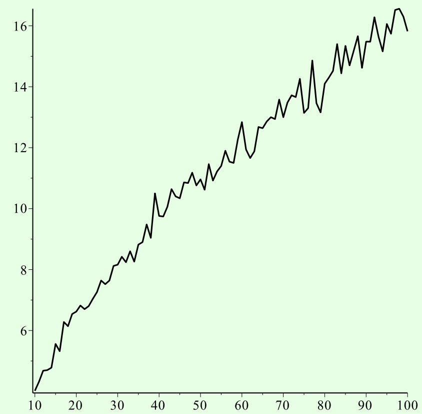


Figure: Average height for 50 randomly generated trees of sizes from 10 to 100

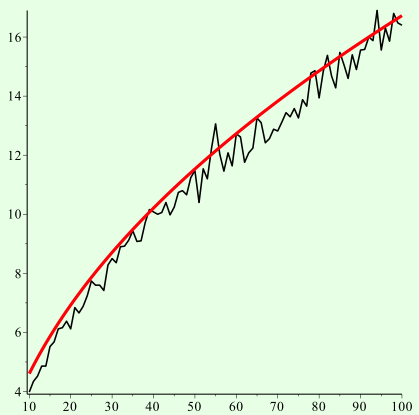


Figure: Average height for 50 randomly generated trees of sizes from 10 to 100 compared to the function $n \mapsto \sqrt{\pi n} - 1$

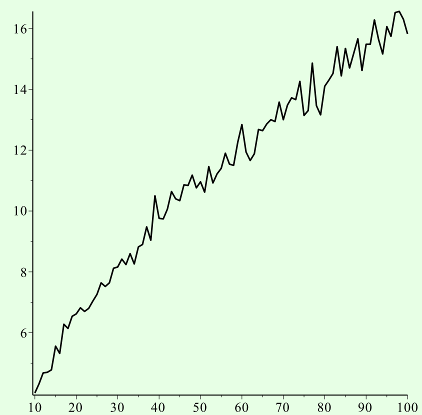


Figure: Aver. aver. level for 50 randomly generated trees of sizes from 10 to 100

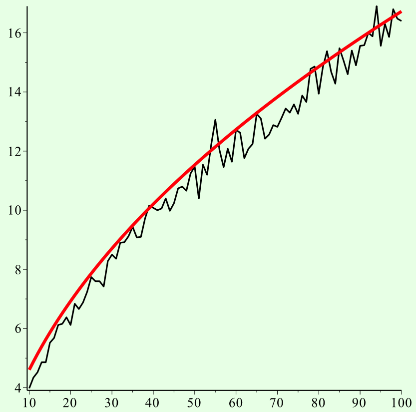


Figure: Aver. aver. level for 50 randomly generated trees of sizes from 10 to 100 compared to the function $n \mapsto \frac{1}{2}(\sqrt{\pi n} - 1)$

Derivation as tree transformation

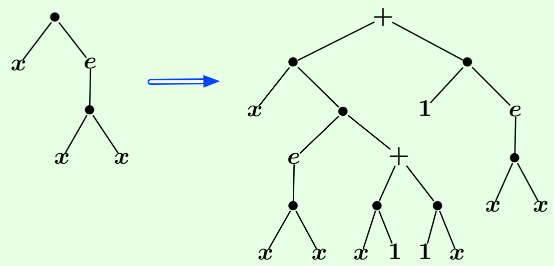


Figure: Derivation of $x \cdot e^{x \cdot x}$

The cost of taking derivatives

- Consider terms (term trees) for simple arithmetic expressions generated by the grammar

$$T \to 0 \,|\, 1 \,|\, x \,|\, a\,T\,T \,|\, m\,T\,T \,|\, e\,T$$

- Symbolic differentiation D w.r.t. x is a term(tree) transformation given by

$0 \to 0$	constant
$1 \to 0$	constant
$x \to 1$	variable
$a\, t_\ell\, t_r \to a\, D(t_\ell)\, D(t_r)$	sum rule
$m\, t_\ell\, t_r \to a\,(m\, t_\ell\, D(t_r))\,(m\, D(t_\ell)\, t_r)$	product rule
$e\, t \to m\,(e\, t)\,(D(t))$	exponent rule

The cost of taking derivatives

- The size $|t|$ of a termtree t is the number of its nodes
- The cost $c_D(t)$ of D executed on a termtree t is $|D(t)|$ so that

$$\begin{aligned} c_D(0) = c_D(1) = c_D(x) &= 1 \\ c_D(a\, t_\ell\ t_r) &= 1 + c_D(t_\ell) + c_D(t_r) \\ c_D(m\, t_\ell\ t_r) &= 3 + |t_\ell| + |t_r| + c_D(t_\ell) + c_D(t_r) \\ c_D(e\, t) &= 2 + |t| + c_D(t) \end{aligned}$$

- Consider now the bivariate generating function

$$c_D(u, z) = \sum_{t \in T} u^{c_D(t)} z^{|t|}$$

- From the cost equations:

$$c_D(u, z) = 3uz + uz\, c_D(u, z)^2 + u^3 z\, c_D(u, uz)^2 + u^2 z\, c_D(u, uz)$$

- There is no hope to solve such an equation explicitly!

The cost of taking derivatives

- One obtains by iteration

$$c_D(u,z) = 3uz + 3u^4z^2 + (9u^3 + 9u^7 + 3u^8)z^3 \\ + (18u^6 + 9u^8 + 18u^{11}9u^{12} + 3u^{13})z^4 + \mathcal{O}(z^5)$$

- Setting $u = 1$ gives the structure generating function

$$t(z) = \sum_{n \geq 0} t_n\, z^n = c_D(1,z) = \frac{1 - z - \sqrt{1 - 2z - 23z^2}}{4z}$$

- This generating function starts as follows:

$$t(z) = 3z + 3z^2 + 21z^3 + 57z^4 + 327z^5 + 1263z^6 + 6753z^7 + \mathcal{O}\left(z^8\right)$$

The cost of taking derivatives

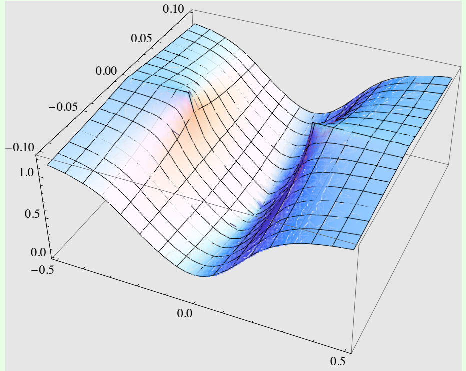


Figure: Plot of the structure generating function $|t(z)|$

Visibly there are two algebraic singularities — that is where $1 - 2z - 23z^2 = 0$ i.e. $z = \frac{1}{23}\left(-1 \pm 2\sqrt{6}\right)$

The cost of taking derivatives

- Knowing $t(z)$ one can solve for the cumulative cost generating function

$$c(z) = \sum_{n \geq 0} c_n\, z^n = \partial_u c_D(u,z)|_{u \leftarrow 1}$$

  where $c_n = \sum_{|t|=n} c_D(t)$
- It turns out that

$$c(z) = \frac{(1 - 2z - 12z^2)\; R - 1 + 3z + 34z^2}{4\, z\, R^2}$$

  where $R = \sqrt{1 - 2z - 23z^2}$
- This generating function starts with

$$c(z) = 3z + 12z^2 + 114z^3 + 525z^4 + 3711z^5 + 19572z^6 \\ + 124194z^7 + 696585z^8 + 4231131z^9 + 24382812z^{10} + \mathcal{O}\left(z^{11}\right)$$

The cost of taking derivatives

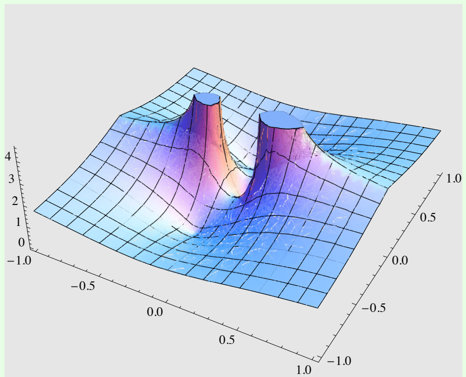


Figure: The cost generation function $c(z)$ and its singularities

Visibly there are poles at precisely those positions where $t(z)$ had algebraic singularities

The cost of taking derivatives

- So what is the asymptotic behaviour of the sequence $\left(\frac{c_n}{t_n}\right)_{n\geq 0}$?
- The series expansions for $t(z)$ and $c(z)$ have the same radius of convergence which is the absolute smallest (dominant) singularity which is

$$\rho = \frac{-1+2\sqrt{6}}{23} \approx 0.169521$$

- Since both series have the same radius of convergence it does not help at all to consider just the exponential growth rate .... one must look closer
- So what is the behaviour of $t(z)$ and of $c(z)$ as $z \to \rho$?

Getting your hands dirty...or your computer busy

$$\begin{aligned}
t(z) = &-\frac{1}{2}\,\frac{\sqrt{6}-12}{2\sqrt{6}-1} \\
&-\frac{1}{2}\,\frac{\sqrt{276-23\sqrt{6}}}{2\sqrt{6}-1}\left(1-\frac{z}{\rho}\right)^{1/2} \\
&+\frac{23}{4}\,\frac{1}{2\sqrt{6}-1}\left(1-\frac{z}{\rho}\right) \\
&+\frac{23}{16}\,\frac{(-71+4\sqrt{6})\sqrt{276-23\sqrt{6}}}{2\sqrt{6}-1}\left(1-\frac{z}{\rho}\right)^{3/2} \\
&+\frac{23}{4}\,\frac{1}{2\sqrt{6}-1}\left(1-\frac{z}{\rho}\right)^{4}+\mathcal{O}\left(\left(1-\frac{z}{\rho}\right)^{5/2}\right) \\
&= 1.2248 - 1.9006\,X + 1.4748\,X^2 - 1.5225\,X^3 + 1.4748\,X^4 + \mathcal{O}(X^5)
\end{aligned}$$

where $X = \sqrt{1-\frac{z}{\rho}}$

Getting your hands dirty...or your computer busy

$$\begin{aligned}
c(z) = &\frac{1}{48}\,\frac{(126+\sqrt{6})\sqrt{6}}{(2\sqrt{6}-1)^2}\left(1-\frac{z}{\rho}\right)^{-1} \\
&-\frac{11}{96}\,\frac{\sqrt{-\frac{2}{23}+\frac{4}{23}\sqrt{6}+46\rho^2}\,(-25+4\sqrt{6})\sqrt{6}}{(2\sqrt{6}-1)^2}\left(1-\frac{z}{\rho}\right)^{-1/2} \\
&-\frac{23}{192}\,\frac{109\sqrt{6}-66}{(2\sqrt{6}-1)^2} \\
&+\frac{1}{4416}\,\frac{-80640+89819\sqrt{6}}{(2\sqrt{6}-1)^2\sqrt{-\frac{2}{23}+\frac{4}{23}\sqrt{6}+46\,\rho^2}}\left(1-\frac{z}{\rho}\right)^{1/2} \\
&+\frac{23}{4608}\,\frac{(-2478+241\sqrt{6})\sqrt{6}}{(2\sqrt{6}-1)^2}\left(1-\frac{z}{\rho}\right)+\mathcal{O}\left(\left(1-\frac{z}{\rho}\right)^{3/2}\right) \\
&= 0.43118\,X^{-2} + 0.36172\,X^{-1} - 3.1019 + 1.6108\,X - 1.5181\,X^2 + \mathcal{O}\left(X^3\right)
\end{aligned}$$

The cost of taking derivatives: the final result

- The asymptotic behaviour turns out to be

$$[z^n]\,t(z) = t_n = \rho^{-n}\left(0.53615\,n^{-3/2} + 0.20105\,n^{-5/2} + \mathcal{O}(n^{-7/2})\right)$$

$$[z^n]\,c(z) = c_n = \rho^{-n}\left(0.43118 + 0.20408\,n^{-1/2} + \mathcal{O}(n^{-3/2})\right)$$

where $\rho^{-1} = 5.89898...$

- So the average case complexity behaves like

$$\frac{c_n}{t_n} \sim \frac{0.43118}{0.53615}\,n^{3/2} = 0.8055...\,n^{3/2}$$

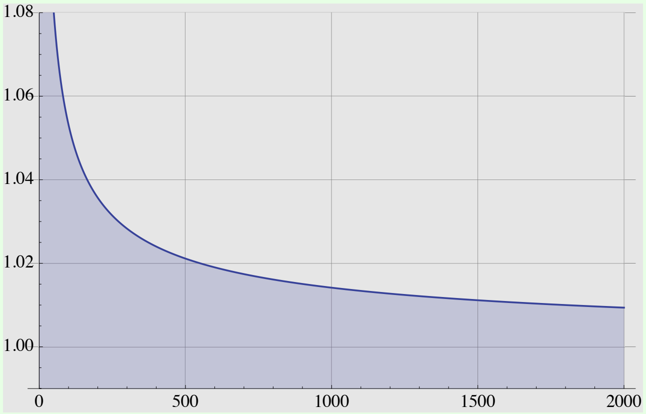


Figure: Plot of $\frac{c_n/t_n}{0.8055 n^{3/2}}$ for $n = 50..2000$

Differentiation with shared subexpressions

- Same setup for term trees and derivation as before: — but now existing subexpresion are not copied, so that
- Symbolic differentiation D

$$
\begin{aligned}
0 \to 0, \quad 1 \to 0, \quad x \to 1 \\
a\, t_\ell\, t_r \to a\, D(t_\ell)\, D(t_r) \\
m\, t_\ell\, t_r \to a\,(m\, t_\ell\, D(t_r))\,(m\, D(t_\ell)\, t_r) \\
e\, t \to m\,(e\, t)\,(D(t))
\end{aligned}
$$

- cost $\widetilde{c}_D(t)$ is now

$$
\begin{aligned}
\widetilde{c}_D(0) = \widetilde{c}_D(1) = \widetilde{c}_D(x) &= 1 \\
\widetilde{c}_D(a\, t_\ell\ t_r) &= 1 + \widetilde{c}_D(t_\ell) + \widetilde{c}_D(t_r) \\
\widetilde{c}_D(m\, t_\ell\ t_r) &= 3 + \mathbf{0} \cdot |\mathbf{t}_\ell| + \mathbf{0} \cdot |\mathbf{t_r}| + \widetilde{c}_D(t_r) + \widetilde{c}_D(t) \\
\widetilde{c}_D(e\, t) &= 2 + \mathbf{0} \cdot |\mathbf{t}| + \widetilde{c}_D(t)
\end{aligned}
$$

Differentiation with shared subexpressions

- The cost generating function

$$\widetilde{c}(u,z) = \sum_{t \in T} u^{\widetilde{c}(t)} z^{|t|}$$

now satisfies

$$\widetilde{c}_D(u,z) = 3uz + uz\,\widetilde{c}_D(u,z)^2 + u^3 z\,\widetilde{c}_D(u,z)^2 + u^2 z\,\widetilde{c}_D(u,z)$$

- The cumulative cost generating function

$$\widetilde{c}(z) = \sum_{n \geq 0} \widetilde{c}_n\, z^n = \partial_u \widetilde{c}_D(u,z)|_{u \leftarrow 1}$$

where $\widetilde{c}_n = \sum_{|t|=n} \widetilde{c}_D(t)$ now starts

$$3\,z + 9\,z^2 + 87\,z^3 + 345\,z^4 + 2403\,z^5 + 11553\,z^6 + 71319\,z^7 + O\left(z^8\right)$$

Differentiation with shared subexpressions

- As before

$$t(z) = 1.2248 - 1.9006\,X + 1.4748\,X^2 - 1.5225\,X^3 + 1.4748\,X^4 + \mathcal{O}(X^5)$$

where $X = \sqrt{1 - \frac{z}{\rho}}$

- But now

$$c(z) = \frac{1}{32} \frac{1}{(1 - 2\sqrt{6})^2\, a\, X} \times$$

$$\times \Big(6608\,\sqrt{6} - 5328 + (368 - 736\sqrt{6})\, a\, X + (12819\,\sqrt{6} - 9894) X^2 + \mathcal{O}($$

where $a = \sqrt{276 - 23\,\sqrt{6}}$

Differentiation with shared subexpressions

- As before, apply the transfer method to obtain
- The asymptotic behaviour turns out to be

$$[z^n]\,t(z) = t_n = \rho^{-n}\left(0.53615\,n^{-3/2} + 0.20105\,n^{-5/2} + \mathcal{O}(n^{-7/2})\right)$$
$$[z^n]\,c(z) = c_n = \rho^{-n}\left(0.84967\,n^{-1/2} - 0.10620\,n^{-3/2} + \mathcal{O}(n^{-5/2})\right)$$

where $\rho^{-1} = 5.89898\ldots$

- So the average case complexity behaves like

$$\frac{c_n}{t_n} \sim \frac{0.84967}{0.53615}\,n = 1.58476\ldots n$$

- Subexpression sharing decreases average case complexity from $\mathcal{O}(n^{3/2})$ to $\mathcal{O}(n)$!
- This hold for large classes of term rewriting algorithms

Meir-Moon's asymptotic counting of trees

- $\Omega = \bigcup_{k\geq 0}\Omega_k$: a set of function symbols of different arities (signature) with $\omega_k = \sharp\Omega_k$
- T_Ω : Ω-(term)-trees so that

$$T_\Omega = \sum_{\omega\in\Omega}\omega.\,T_\Omega^{ar(\omega)} = \sum_{k\geq 0}\sum_{\omega\in\Omega_k}\omega.\,T_\Omega^k$$

- $T_{\Omega,n}$: Ω-(term)-trees of size n, $\quad t_{\Omega,n} = \sharp T_{\Omega,n}$
- Theorem: (under mild technical conditions)

$$t_{\Omega,n} = \sqrt{\frac{\omega(\tau)}{2\pi\,\omega''(\tau)}}\,\rho^{-n}\;n^{-3/2}\left(1 + O(\frac{1}{n})\right)$$

where $\omega(z) = \sum_k \omega_k\,z^k$ and
  - τ is the smallest positive root of $\omega(z) = z\,\omega'(z)$
  - $\rho = \tau/\omega(\tau)$

Meir-Moon's asymptotic counting of trees

- Some remarks about the proof
  - The generating function

$$t_\Omega(z) = \sum_{t\in T_\Omega} z^{\mathtt{size}(t)} = \sum_{n\geq 0} t_{\Omega,n}\;z^n$$

satisfies (uniquely) the fixed point equation

$$y(z) = z\cdot\omega(y(z))$$

  - The Implicit Function Theorem gives information about the existence of a unique analytic solution in the vicinity of $z = \rho$

Meir-Moon's asymptotic counting of trees

- Some remarks about the proof (contd.)
  - In the vicinity of $z = \rho$

$$t_\Omega(z) = g(z) + h(z)\sqrt{1 - \frac{z}{\rho}}$$

with analytic functions $g(z), h(z)$ (around ρ) which satisfy

$$h(\rho) = \tau \text{ and } g(\rho) = -\sqrt{\frac{2\omega(\tau)}{\omega''(\tau)}}$$

  - Under appropriate technical conditions the (dominant) singularity of $t_\Omega(z)$ at $z = \rho$ is well-behaved (is a "Camembert-singularity") – thus the Transfer Principle can be applied

Average level and height of ordered trees

- The `average level` (or `pathlength`) of ordered trees can be obtained
  1. by a similar technique as used for evaluation of symbolic differentiation....
  2. by an argument that employs Lagrange's formula...
- There seems to be an intimate relation between average height and average level ...
  1. This is somewhat surprising!
  2. So why is that indeed the case?

The number of ordered trees

- Counting ordered trees is easy!
  - Let t_n = the number of ordered trees with n nodes
  - Let $t(z) = \sum_{n\geq 0} t_n z^n$ be the generating function

$$t(z) = z + z^2 + 2z^3 + 5z^4 + 14z^5 + 42z^6 \cdots$$

  Catalan numbers show up again!
  - From the structure of ordered trees

$$\begin{aligned} t(z) &= \underbrace{z}_{\substack{\uparrow\\ \text{root}}} \cdot \big(\underbrace{1}_{\substack{\uparrow\\ \text{no}\\ \text{subtree}}} + \underbrace{t(z)}_{\substack{\uparrow\\ \text{one}\\ \text{subtree}}} + \underbrace{t(z)^2}_{\substack{\uparrow\\ \text{two}\\ \text{subtrees}}} + \underbrace{t(z)^3}_{\substack{\uparrow\\ \text{three}\\ \text{subtrees}}} + \cdots \big) \\ &= z \cdot \frac{1}{1-t(z)} \end{aligned}$$

  - and thus

$$t(z) = \frac{1-\sqrt{1-4z}}{2} \qquad t_n = \frac{1}{n}\binom{2n-2}{n-1} = d_{n-1}$$

Level distribution of ordered trees

- Comment: There is a neat correspondence between binary trees with n internal nodes and ordered trees with $n+1$ nodes (and Dyck words of length $2n$), which has been studied and used a lot ...

  ... but `height` and `pathlength` do not behave well w.r.t. to it
- Consider now

$$\ell_{n,k} = \text{total number of nodes on level } k \text{ in all trees of size } n$$

- This quantity has a nice expression

$$\ell_{n,k} = \frac{2k+1}{n+k}\binom{2n-2}{n-k-1}$$

  In particular: $\ell_{n,0} = t_n = d_{n-1}$

Level distribution of ordered trees

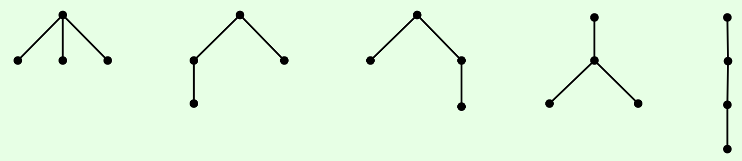


Figure: Ordered trees of size 4

level distribution: $\ell_{4,0} = 5, \ell_{4,1} = 9, \ell_{4,2} = 5, \ell_{4,3} = 1$
cumulated pathlength:

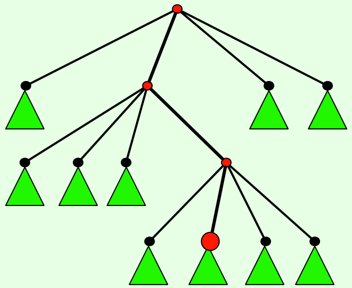
$$3+4+4+5+6 = 22 = 0\cdot\ell_{4,0} + 1\cdot\ell_{4,1} + 2\cdot\ell_{4,2} + 3\cdot\ell_{4,3}$$

A closer combinatoriel look

- Determining $\ell_{n,k}$ via generating functions
  - Claim: $\ell_{n,k}$ is the coefficient of z^n in the series expansion (around $z=0$) of the function

$$z^k \cdot t(z) \cdot \frac{1}{(1-t(z))^{2k}}$$

  - The explanation (case $k=3$):



from the very early days of complex analysis ... Lagrange!

- A version of Lagranges's formula
  - Let $\phi(z)$ be a known “analytic” function around $z=0$ with $\phi(0) \neq 0$
  - Let $w(z)$ be the “analytic” function defined implicitly by

$$w(z) = z \cdot \phi(w(z))$$

    (implicit function theorem!)
  - Then:

$$\text{coefficient of } z^n \text{ in the series expansion of } w(z)^k$$
$$=$$
$$\frac{k}{n} \cdot \text{coefficient of } z^{n-1} \text{ in the series expansion of } z^{k-1} \cdot \phi(z)^n$$

  - shorthand:

$$[z^n]\, w(z)^k = \frac{k}{n}\, [z^{n-1}]\, z^{k-1} \cdot \phi(z)^n$$

- This is residue calculus + variable transform
- This helps, if $\phi(z)$ is sufficiently simple ...

now let's calculate ...

- For ordered trees: $t(z) = \frac{z}{1-t(z)}$, so $\phi(z) = \frac{1}{1-z}$
- Let's go ...

$$\begin{aligned}
\ell_{n,k} &= [z^n]\, z^k \cdot t(z) \cdot \frac{1}{(1-t(z))^{2k}} && \text{the tree decomposition} \\
&= [z^n]\, z^k \cdot t(z)^{2k+1} && \text{using } t(z) = \frac{1}{1-t(z)} \\
&= [z^{n+k}]\, t(z)^{2k+1} && \text{just shifting} \\
&= \frac{2k+1}{n+k} \cdot [z^{n+k-1}]\, z^{2k} \cdot \frac{1}{(1-z)^{n+k}} && \text{Lagrange strikes with } \phi(z)=\tfrac{1}{1-z} \\
&= \frac{2k+1}{n+k} \cdot [z^{n-k-1}]\, \frac{1}{(1-z)^{n+k}} && \text{shifting again} \\
&= \frac{2k+1}{n+k} \binom{2n-2}{n-k-1} && \text{Newtons binomial theorem}
\end{aligned}$$

Level distribution: the result

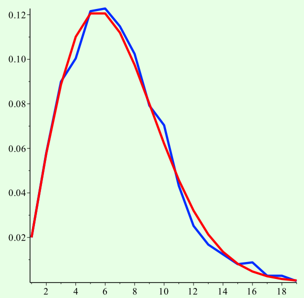
- Consequence:
  - By Stirling's formula one obtains for the average number of nodes on level k in trees of size n:

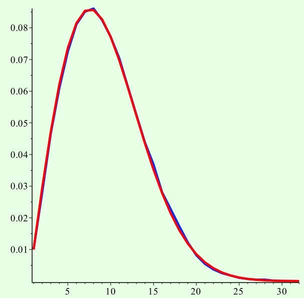
$$\overline{\ell_{n,k}} = \frac{\ell_{n,k}}{t_n} = \frac{2k+1}{n+k} \cdot \frac{\binom{2n-2}{n-k-1}}{n \cdot \binom{2n-2}{n-1}} \sim 2\,k\,\mathrm{e}^{-k^2/n}$$

    (at least if $k \approx \sqrt{n}$)
  - Put $k = \lambda\sqrt{n}$, then

$$\overline{\ell_{n,\lambda\sqrt{n}}} \sim 2\,\lambda\,\sqrt{n}\,\mathrm{e}^{-\lambda^2}$$

    and this achieves its maximum (for n fixed) at $\lambda = 1/2$





Sampled profiles (blue) for ordered trees, compared to true average (red), 50 samples for $n = 50$ (left), 500 samples for $n = 100$(right)

more juggling with generating series...

- The cumulated pathlength for trees of size n is (with $t \equiv t(z)$)

$$
\begin{aligned}
\sum_k k\,\ell_{n,k} &= \sum_k k\,[z^n]\,z^{-k}t^{2k+1} && \text{decomposition}\\
&= [z^n]\sum_k k\,z^{-k}t^{2k+1} && \text{linearity}\\
&= [z^n]\,t\cdot\sum_k k\left(t^2/z\right)^k && \text{rearranging}\\
&= [z^n]\,t\cdot\frac{t^2/z}{(1-t^2/z)^2} && \text{derivative of geometric series}\\
&= [z^n]\,z\,t\cdot\frac{t^2}{(z-t^2)^2} && \text{rearranging}\\
&= [z^{n-1}]\,\frac{t}{(1-2t)^2} && \text{using } z-t^2=t-2t^2\\
&= \tfrac{1}{2}\left(4^{n-1}-\tbinom{2n-2}{n-1}\right) && \text{using } 1-2t=\sqrt{1-4z}
\end{aligned}
$$

Average level: the result

- Consequence:
  - The average `level` of nodes in ordered trees of size n is

$$
\frac{\sum_k k\cdot\ell_{n,k}}{n\cdot t_n} = \frac{\frac{1}{2}\left(4^{n-1}-\binom{2n-2}{n-1}\right)}{n\cdot\frac{1}{n}\binom{2n-2}{n-1}} \quad \sim_{n\to\infty} \quad \frac{1}{2}\sqrt{\pi n}-\frac{1}{2}+\mathcal{O}(n^{-1/2})
$$

Height vs. pathlength

- What about `height`?
- `height` is an important parameter, but difficult to treat, because

$$
\mathtt{height}(tree) = 1 + \mathtt{max}_{t\in subtrees(tree)}\mathtt{height}(t)
$$

  and `max` is a nonlinear function!
- Compare `pathlength`:

$$
\mathtt{pathlength}(tree) = \sum_{t\in subtrees(tree)} \mathtt{pathlength}(t) + \mathtt{size}(t)
$$

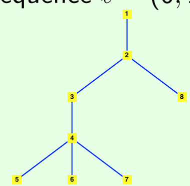
  which is linear (additive)

The fundamental results

- (de Bruijn-Knuth-Rice, 1972)
  The average height of ordered trees with n nodes behaves as
  $$\sim_{n\to\infty} \sqrt{\pi n}$$
- (Flajolet-Odlyzko, 1982)
  The average height of binary trees with n internal nodes behaves as
  $$\sim_{n\to\infty} 2\sqrt{\pi n}$$

... a completely different approach ... level sequences

- The result for the height of ordered trees can be obtained combinatorially (no complex analysis needed !!) from the above result about the average level
- Ordered trees of size n can be represented by level sequences
  $$t \mapsto \ell(t) = (\ell_1, \ell_2, \ldots, \ell_n)$$
  where $\ell_1 = 0, \quad 0 < \ell_{j+1} \le \ell_j + 1 \;\; (1 \le j < n)$
  (recording node levels in preorder traversal)
  L_n : level sequences of length n
- Example: Tree with level sequence $\ell = (0,1,2,3,4,4,4,2)$



level sequences of ordered trees

- `height` and `pathlength` translate easily
  $$\mathtt{height}(t) = \max_{1\le j\le n} \ell_j =: \mu(\ell)$$
  $$\mathtt{pathlength}(t) = \ell_1 + \ell_2 + \cdots + \ell_n =: \lambda(\ell)\cdot n$$
  So $\lambda(\ell)$ is the average level in t
- Interesting fact:
  - There exists an involution $\ell \mapsto \widetilde{\ell}$ on L_n which satisfies
  $$\lambda(\ell) + \lambda(\widetilde{\ell}) - 1 < \frac{\mu(\ell) + \mu(\widetilde{\ell})}{2} \le \lambda(\ell) + \lambda(\widetilde{\ell})$$
  - This is tricky, as it requires an extension of the concept of level sequences to sequences which do no longer correspond to trees ...
- Consequence (by averaging over L_n):
  $$\overline{\mu}_n \approx 2\,\overline{\lambda}_n \sim_{n\to\infty} 2\sqrt{\pi n}$$

level sequences of ordered trees and more

- Generalized level sequences of length n are sequences $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$ such that
  $$\ell_1 \le 0, \quad \ell_n \ge 0, \quad \ell_{j+1} \le \ell_j + 1 \;\; (1 \le j < n)$$
  GL_n : generalized level sequences of length n, $\quad \sharp GL_n = \binom{2n-1}{n}$
- Shifting generalized level sequences
  $$\sigma(\ell_1, \ell_2, \ldots, \ell_n) \mapsto \begin{cases} (\ell_2 - 1, \ldots, \ell_n - 1, 0) & \text{if } \ell_1 = 0 \\ (\ell_1 + 1, \ell_2 + 1, \ldots, \ell_n + 1) & \text{if } \ell_1 < 0 \end{cases}$$
- Facts:
  - GL_n decomposes into σ-orbits of length $2n-1$
  - Each σ-orbit contains exactly one level sequence (alias tree!)

Example of a σ-orbit in LS_6

0	**1**	**2**	**1**	**2**	**3**	$\in L_6$
0	1	0	1	2	0	
0	−1	0	1	−1	0	
−2	−1	0	−2	−1	0	
−1	0	1	−1	0	1	$\downarrow \sigma$
0	1	2	0	1	2	
0	1	−1	0	1	0	
0	−2	−1	0	−1	0	
−3	−2	−1	−2	−1	0	
−2	−1	0	−1	0	1	
−1	0	1	0	1	2	

Two σ-orbits in GL_6 related by the reflection

$$\rho : (\ell_1, \ell_2, \ldots, \ell_n) \mapsto (-\ell_n, \ldots, -\ell_2, -\ell_1)$$

and the definition of $\ell \mapsto \widetilde{\ell}$

ℓ	**0**	**1**	**2**	**1**	**2**	**1**	−1	−2	−1	−2	−1	0	
	0	1	0	1	0	0	0	0	−1	0	−1	0	
	0	−1	0	−1	−1	0	0	1	1	0	1	0	
	−2	−1	−2	−2	−1	0	**0**	**1**	**2**	**2**	**1**	**2**	$\widetilde{\ell}$
	−1	0	−1	−1	0	1	−1	0	1	1	0	1	
$\downarrow \sigma$	0	1	0	0	1	2	−2	−1	0	0	−1	0	$\uparrow \sigma$
	0	−1	−1	0	1	0	0	−1	0	1	1	0	
	−2	−2	−1	0	−1	0	0	1	0	1	2	2	
	−1	−1	0	1	0	1	−1	0	−1	0	1	1	
	0	0	1	2	1	2	−2	−1	−2	−1	0	0	
	−1	0	1	0	1	0	0	−1	0	−1	0	1	

Balancing trees makes the analysis difficult

- Consider now the familiar balanced 2-3 trees:

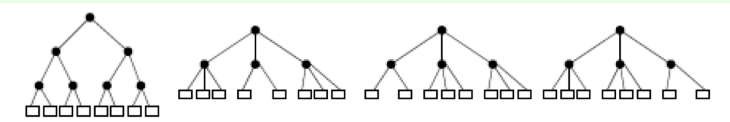


Figure: Balanced 2-3-trees with 8 leaves

- Internal nodes have 2 or 3 successors
- All leaves on the same height
- `size` is the number of leaves
- e_n = number of balanced 2-3 trees of size n
- First values

n	0	1	2	3	4	5	6	7	8	9	10
e_n	0	1	1	1	1	2	2	3	4	5	8

Apparent simplicity can fool you...

- Sequence $(e_n)_{n\geq 0}$ seems to grow slowly but

$$e_{100} = 5520498313790316062$$

- So how fast does this sequence really grow?
- Generating function $e(z) = \sum_{n\geq 0} e_n z^n$ satisfies

$$e(z) = z + e(z^2 + z^3)$$

- which is equivalent to

$$e_n = \sum_{k=0}^{n} \binom{k}{n-2k} e_k, \quad e_0 = 0, e_1 = 1$$

- looks "innocuous", but isn't!

Digging deeper — and finding ϕ again

- Let $\sigma(z) = z^2 + z^3$ and consider the sequence by composition

$$\sigma^{(t+1)}(z) = \sigma(\sigma^{(t)}(z)) \quad \sigma^{(0)}(z) = z$$

- Then by unfolding the fixed-point equation

$$e(z) = \sigma^{(0)}(z) + \sigma^{(1)}(z) + \sigma^{(2)}(z) + \sigma^{(3)}(z) + \cdots$$

- The equation $\sigma(z) = z$ has $\rho = \phi^{-1}$ as unique positive fixed point
- Easy exercise: $\left(\sigma^{(n)}(z)\right)_{n\geq 0} \to 0$ (rapidly) for any $z \in \mathbb{C}$ with $|z| < \rho$
- Easy exercise: $e(z)$ is unbounded as $z \to \rho^-$
- $\rho = \phi^{-1}$ is the radius of convergence of $e(z)$ hence:

$$e_n \asymp \left(\frac{1+\sqrt{5}}{2}\right)^n$$

For more information: work harder!

- But what about the subexponential factor?
- Needs analysis using the nature of the singularity
- On gets

$$e_n = \frac{\phi^n}{n}\,\Omega(\log n) + \mathcal{O}(\frac{\phi^n}{n^2})$$

where $\Omega(z)$ is a periodic function with mean $\phi \log(4-\phi) \approx 0.71208$ and period $\log(4-\phi) \approx 0.86792$

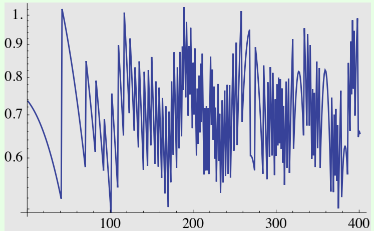


Figure: Plot of $e_n/(\phi^n/n)$ for $n = 1..400$ in logarithmic scale

The fractal nature of convergence

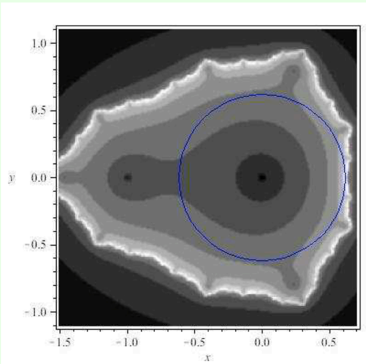


Figure: Domain of "analyticity" and circle of convergence of $e(z)$

Picture taken from the "definitive" book *Analytic Combinatorics* by Ph. Flajolet and R. Sedgewick, Cambridge UP, 2009.