Catalan numbers

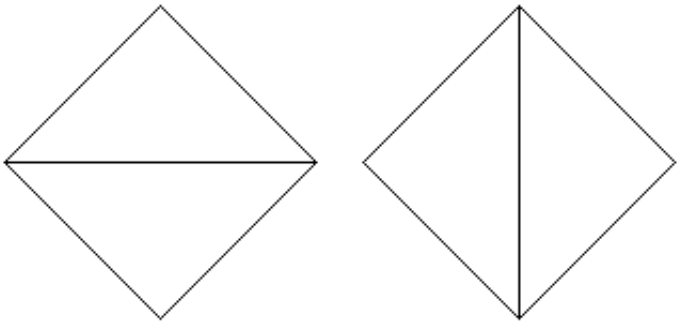
The Catalan numbers (1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, ...), named after Eugène Charles Catalan (1814--1894), arise in a number of problems in combinatorics. They can be computed using this formula:

$$\frac{\binom{2n}{n}}{n+1}$$

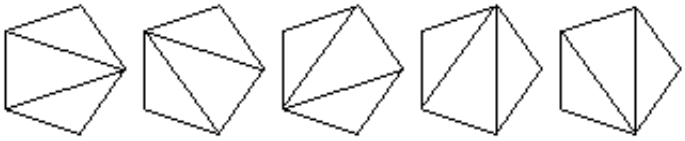
Among other things, the Catalan numbers describe the number of ways a polygon with $n+2$ sides can be cut into n triangles, the number of ways in which parentheses can be placed in a sequence of numbers to be multiplied, two at a time; the number of rooted, trivalent trees with $n+1$ nodes; and the number of paths of length $2n$ through an n-by-n grid that do not rise above the main diagonal.

Polygon diagrams (code hidden for your protection):

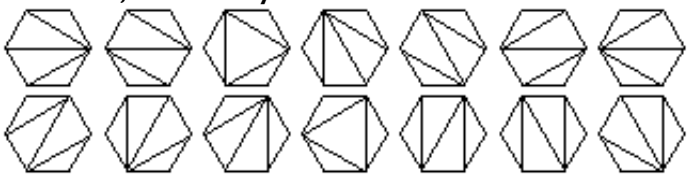
4 sides, 2 ways:



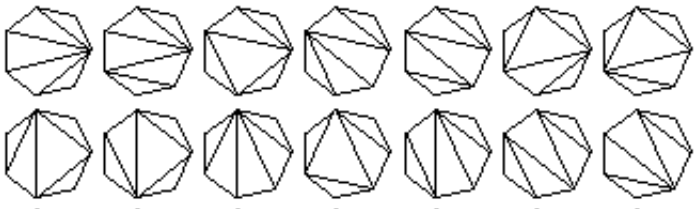
5 sides, 5 ways:

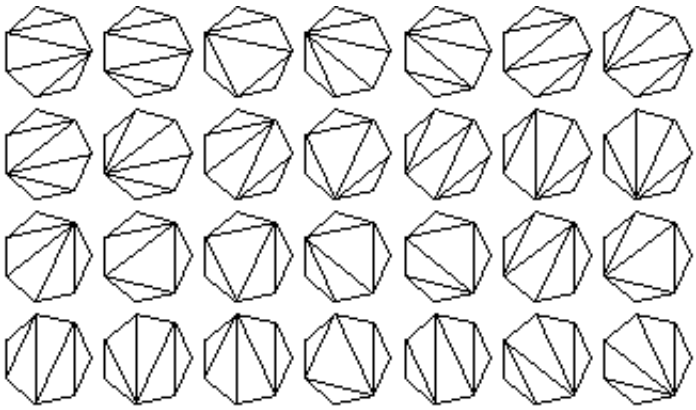


6 sides, 14 ways:

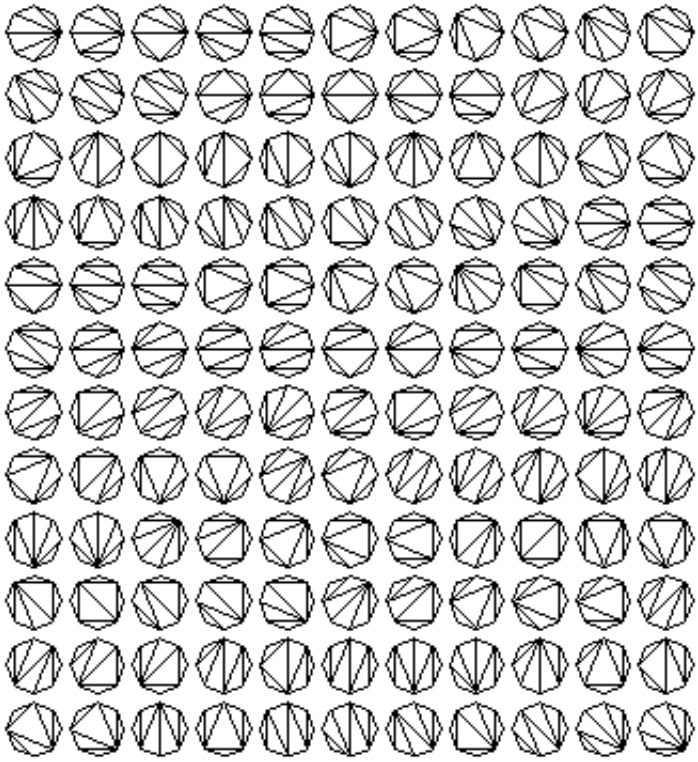


7 sides, 42 ways:





8 sides, 132 ways:



9 sides, 429 ways:
(Hidden in file catalan9.gif; around 29K.)

Multiplication diagrams:

3 numbers:

```
(1 (2 3))    ((1 2) 3)
```

4 numbers:

```
(1 (2 (3 4)))    (1 ((2 3) 4))
((1 2) (3 4))    ((1 (2 3)) 4)
(((1 2) 3) 4)
```

5 numbers:

```
(1 (2 (3 (4 5))))    (1 (2 ((3 4) 5)))
```

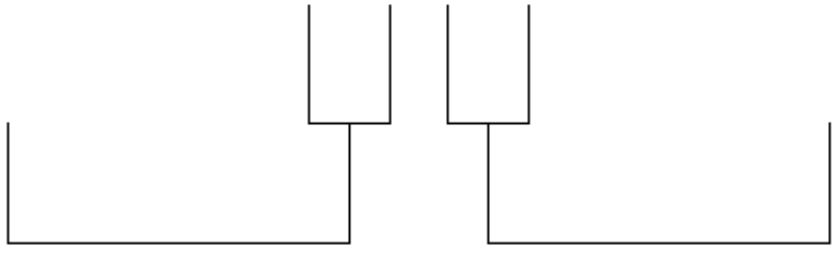
```
(1 ((2 3) (4 5)))    (1 ((2 (3 4)) 5))
(1 (((2 3) 4) 5))    ((1 2) (3 (4 5)))
((1 2) ((3 4) 5))    ((1 (2 3)) (4 5))
((1 (2 (3 4))) 5)    ((1 ((2 3) 4)) 5)
(((1 2) 3) (4 5))    (((1 2) (3 4)) 5)
(((1 (2 3)) 4) 5)    ((((1 2) 3) 4) 5)
```

6 numbers:

```
(1 (2 (3 (4 (5 6)))))    (1 (2 (3 ((4 5) 6))))
(1 (2 ((3 4) (5 6))))    (1 (2 ((3 (4 5)) 6)))
(1 (2 (((3 4) 5) 6)))    (1 ((2 3) (4 (5 6))))
(1 ((2 3) ((4 5) 6)))    (1 ((2 (3 4)) (5 6)))
(1 ((2 (3 (4 5))) 6))    (1 ((2 ((3 4) 5)) 6))
(1 (((2 3) 4) (5 6)))    (1 (((2 3) (4 5)) 6))
(1 (((2 (3 4)) 5) 6))    (1 ((((2 3) 4) 5) 6))
((1 2) (3 (4 (5 6))))    ((1 2) (3 ((4 5) 6)))
((1 2) ((3 4) (5 6)))    ((1 2) ((3 (4 5)) 6))
((1 2) (((3 4) 5) 6))    ((1 (2 3)) (4 (5 6)))
((1 (2 3)) ((4 5) 6))    ((1 (2 (3 4))) (5 6))
((1 (2 (3 (4 5)))) 6)    ((1 (2 ((3 4) 5))) 6)
((1 ((2 3) 4)) (5 6))    ((1 ((2 3) (4 5))) 6)
((1 ((2 (3 4)) 5)) 6)    ((1 (((2 3) 4) 5)) 6)
(((1 2) 3) (4 (5 6)))    (((1 2) 3) ((4 5) 6))
(((1 2) (3 4)) (5 6))    (((1 2) (3 (4 5))) 6)
(((1 2) ((3 4) 5)) 6)    (((1 (2 3)) 4) (5 6))
(((1 (2 3)) (4 5)) 6)    (((1 (2 (3 4))) 5) 6)
(((1 ((2 3) 4)) 5) 6)    ((((1 2) 3) 4) (5 6))
((((1 2) 3) (4 5)) 6)    ((((1 2) (3 4)) 5) 6)
((((1 (2 3)) 4) 5) 6)    (((((1 2) 3) 4) 5) 6)
```

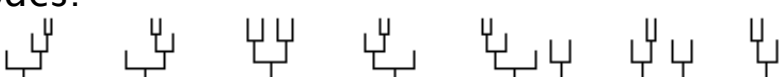
Tree diagrams:

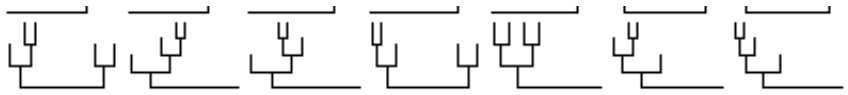
3 nodes:



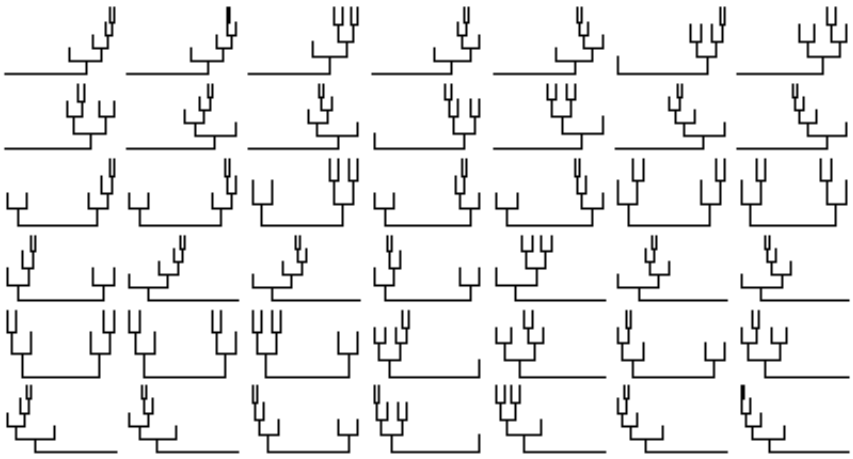
4 nodes:

5 nodes:

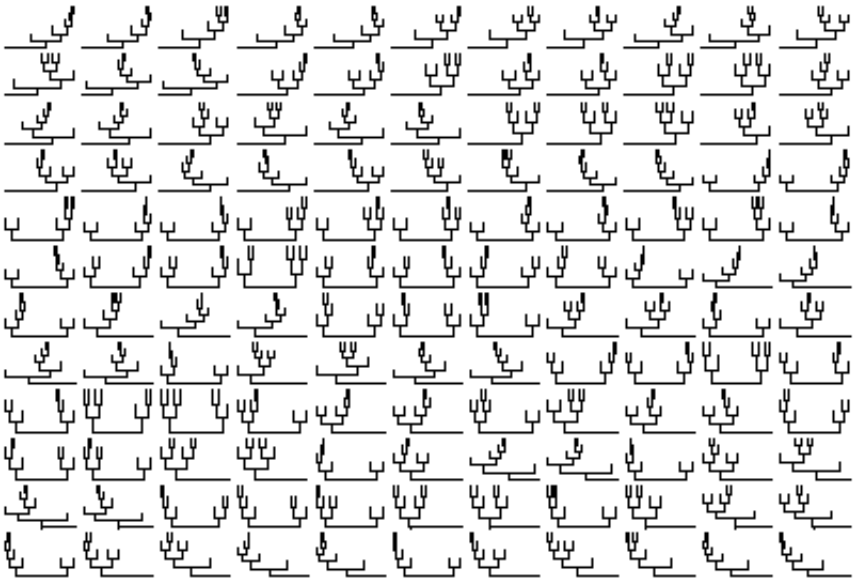




6 nodes:



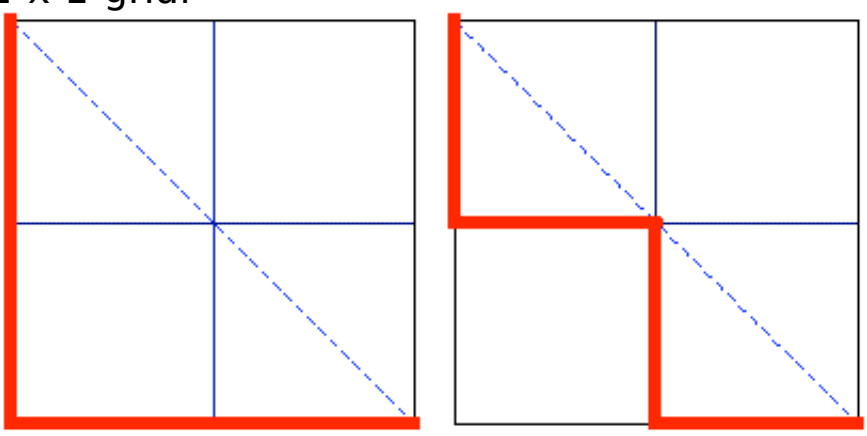
7 nodes:



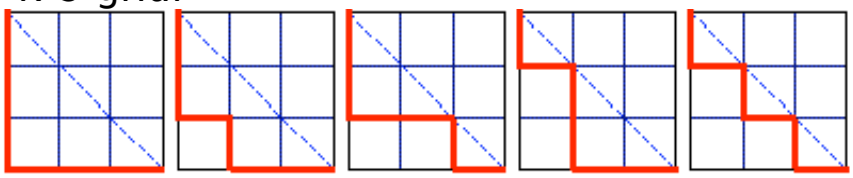
8 nodes:
(Tucked away in file cattree8.gif; around 17K.)

Path diagrams:

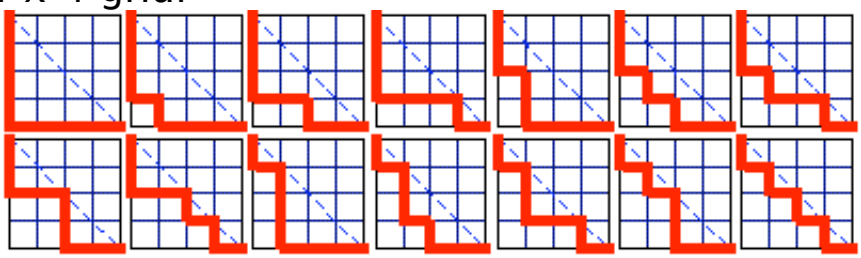
2 x 2 grid:



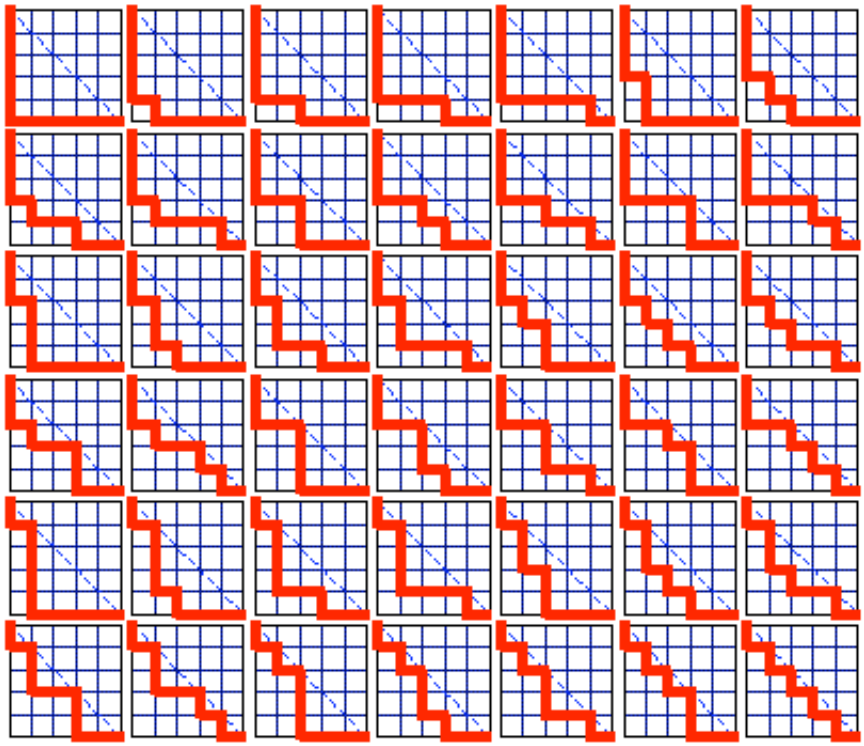
3 x 3 grid:



4 x 4 grid:



5 x 5 grid:



6 x 6 grid:
(Out of the way in file catalanpath6.gif; around 24K.)

Designed and rendered using *Mathematica* 3.0 for the Apple Macintosh. Inspiration and facts (though not figures) by Brian Hayes, "A Question of Numbers", *American Scientist*, January–February 1996; Steven S. Skiena, *Implementing Discrete Mathematics: Combinatorics and Graph Theory with Mathematica*, Addison-Wesley, 1990; Fred S. Roberts, *Applied Combinatorics*, Prentice-Hall, 1984; and D. E. Knuth, *Sorting and Searching* (vol. 3 of *The Art of Computer Programming*), Addison-Wesley, 1973. Catalan dates from Florian Cajori, *A History of Mathematics*, The Macmillan Company, 1922.

See also Martin Gardner, *Time Travel and Other Mathematical Bewilderments*, chapter 20, W. H. Freeman, 1988; and Ilan Vardi, *Computational Recreations in Mathematica*, chapter 9, Addison-Wesley, 1991.

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