1. $k > \ell \geq 0 \Rightarrow n^\ell \in o(n^k)$

$$\lim_{n\to\infty} \frac{n^\ell}{n^k} = \lim_{n\to\infty} \frac{1}{n^{k-\ell}} = 0$$

2. $k > \ell \geq 0 \Rightarrow n^k + n^\ell \in \Theta(n^k)$

$$n^k \leq n^k + n^\ell \quad \Rightarrow \quad n^k + n^\ell \in \Omega(n^k)$$

$$n^k + n^\ell \leq 2 \cdot n^k \quad \Rightarrow \quad n^k + n^\ell \in O(n^k)$$

3. $p(n) = \sum_{i=0}^{k} p_i\, n^i$ mit $p_k \neq 0 \Rightarrow p(n) \in O(n^k)$

$$\begin{aligned}
|p(n)| &= \left|\sum_{i=0}^{k} p_i\, n^i\right| \\
&\leq \sum_{i=0}^{k} |p_i|\, n^i \\
&\leq \max_{0\leq i\leq k} \left(|p_i|\right) \cdot \sum_{0\leq i\leq k} n^i \\
&\leq \max_{0\leq i\leq k} \left(|p_i|\right) \cdot \frac{n^{k+1}-1}{n-1} \\
&\leq \max_{0\leq i\leq k} \left(|p_i|\right) \cdot 2 \cdot n^k \quad (\text{falls } n \geq 2) \\
&\Rightarrow \quad p(n) \in O(n^k)
\end{aligned}$$

4. $p(n) = \sum_{i=0}^{k} p_i\, n^i$ mit $p_k > 0 \Rightarrow p(n) \in \Omega(n^k)$

$$\begin{aligned}
p(n) &= p_k\, n^k + \sum_{i=0}^{k-1} p_i\, n^i \\
&= p_k\, n^k \left(1 + \frac{1}{n^k} \sum_{i=0}^{k-1} \frac{p_i}{p_k}\, n^i\right) \\
&\geq p_k\, n^k \left(1 - \frac{1}{n^k} \sum_{i=0}^{k-1} \left|\frac{p_i}{p_k}\right| n^i\right) \\
&\geq p_k\, n^k \left(1 - \frac{1}{n^k} \cdot \max_{0\leq i<k} \left(\left|\frac{p_i}{p_k}\right|\right) \cdot 2 \cdot n^{k-1}\right) \\
&= p_k\, n^k \left(1 - 2 \cdot \frac{\max}{n}\right) \\
&\geq \frac{1}{2}\, p_k\, n^k \quad \text{für } n \geq 4 \cdot \max \\
&\Rightarrow \quad p(n) \in \Omega(n^k)
\end{aligned}$$

5. $n^k = o(2^n)$ $(k \in \mathbb{N})$

$$\forall n, \ell \in \mathbb{N} \quad \ln n \leq \frac{n}{\ell} + \ell$$

$$\Downarrow$$

$$\lim_{n\to\infty} \big(n - k \cdot \underbrace{\log n}_{\leq 2\ln n}\big) \geq \lim_{n\to\infty} \left(n - 2k\left(\frac{n}{4k} + 4k\right)\right) = \lim_{n\to\infty} \left(\frac{n}{2} - 8k^2\right) = \infty$$

$$\Downarrow$$

$$\lim_{n\to\infty} \frac{n^k}{2^n} = \lim_{n\to\infty} 2^{k\cdot\log n - n} = 0$$

6. $\log^k n = o(n^\epsilon)$ $(\epsilon > 0, k \in \mathbb{N})$

Fall $k = 1$

$$\lim_{n\to\infty} \frac{\log n}{n^\epsilon} = \lim_{n\to\infty} \frac{1/(n \cdot \ln 2)}{\epsilon \cdot n^{\epsilon-1}} = \frac{1}{\ln 2 \cdot \epsilon} \lim \frac{1}{n^\epsilon} = 0$$

Fall $k > 1$ (Induktion)

$$\lim_{n\to\infty} \frac{\log^k n}{n^\epsilon} = \frac{1}{\ln 2} \lim_{n\to\infty} \frac{k \cdot \log^{k-1} n \cdot \frac{1}{n}}{\epsilon \cdot n^{\epsilon-1}} = \frac{k}{\epsilon \cdot \ln 2} \cdot \lim_{n\to\infty} \frac{\log^{k-1} n}{n^\epsilon} = 0$$

7. $2^n \in o(2^{2n})$ (aber: $n \notin o(2n)$)

$$\lim_{n\to\infty} \frac{2^n}{2^{2n}} = \lim \frac{1}{2^n} = 0$$

8. O vs. Ω, o vs. ω

$$f \in O(g) \Leftrightarrow g \in \Omega(f)$$
$$f \in o(g) \Leftrightarrow g \in \omega(f)$$

9. Transitivität $(\mathcal{O} \in \{O, o, \Theta, \Omega, \omega\})$

$$f \in \mathcal{O}(g) \wedge g \in \mathcal{O}(h) \Rightarrow f \in \mathcal{O}(h)$$

10. Additivität $(\mathcal{O} \in \{O, o, \Theta, \Omega, \omega\})$

$$f \in \mathcal{O}(h) \wedge g \in \mathcal{O}(h) \Rightarrow f + g \in \mathcal{O}(h)$$