

## INTENSIONAL AND MODAL LOGIC

Why do we consider extensions to the standard logical language(s)?

⇒ Requirements of knowledge representation / domain modelling

- Intensional expressions: Create contexts which violate standard principles of logic, e.g. substitution of identities
- Modalities:
  - Necessity and possibility
  - Epistemic operators (knowledge and belief)
- Temporal logic
- Intensional logic with types for natural language semantics: Montague

## Meaning and Reference

FREGE: Predicators (general terms) have

- “Sinn” = **meaning** (sense, intension, compréhension) vs.
- “Bedeutung” = **reference** (denotation, extension, étendue)

Example: “*The morning star is identical with the evening star.*”

- **Reference** of a name (nominator) is the named object,
- **meaning** is the mode “how the object is given”

... extension to predicates and sentences

CARNAP:

	name	predicate	sentence
intension	indiv. concept	concept	proposition
extension	object	class	truth value

## Intensional Logic

**Intensional logic:** Theory of logical systems, where there are expressions, whose extension is not uniquely defined by the extensions of their subexpressions, but by their intensions.

⇒ The truth value of logically composed expressions is not simply a logical function of the truth values of their subexpressions.

*Examples:* Sentences containing “always” indicate an intensional context. In general: “indexical expressions”

“The president of the Federal Republic of Germany is always the first state representative.”

and “Horst Koehler is president of the Federal Republic of Germany.”

⇒ “Horst Koehler is always the first state representative.”

Intensional logics are systems that distinguish an expression's **intension** (i.e. sense, meaning) from its **extension** (reference, denotation).

Reason: Capture the logical "behaviour" of intensional expressions, which create contexts which violate standard principles of logic, most notably the law of substitution of identities:

"From  $a = b$  and  $P(a)$  it follows that  $P(b)$ "

⇒ Intensional logics provide an analysis of meaning.

Another example: "obviously"

$Scott = the\ author\ of\ Waverley$  and  $obviously\ Scott = Scott$ ,  
but **not**  $obviously\ Scott = the\ author\ of\ Waverley$ .

Modality: If  $A$  and  $B$  are both true, "it is necessary that  $A$ " may have a truth value different from "it is necessary that  $B$ ".

E.g., if  $A \Leftrightarrow "7 = \text{the number of world miracles}"$  ( $A$  is a contingent truth), and  $B \Leftrightarrow "7 = 7"$ . (⇒ Modal Logic)

## Examples of Intensional Logics

- Modal logic (ontic, deontic, epistemic)
- Temporal logic (mellontic)
- Relevance logic, logic of "strict implication", "logic of entailment"
- Counterfactual conditionals

Intensional contexts are signalled by, e.g.

- "It is necessary, that. . ." (ontic modal logic)
- "Thou shalt. . ." (deontic modal logic)
- " $x$  believes, that. . ." (epistemic modal logic)

- "Always. . ." (temporal logic)
- ". . . follows from. . ." (logic of entailment, relevance logic)
- "If Rattle had been conducting, I would have gone to the concert."  
(counterfactual conditionals)

## Intensions and "Possible Worlds"

Traditional viewpoint: An extensional interpretation (as for FOL) fixes the truth values of sentences in our world.

CARNAP's idea: Generalize interpretations in a way that they fix the truth values of sentences in all "*possible worlds*".

**Possible world:** Possible states (state of affairs) and state combinations s.t. every complete state description describes a possible world.

"Complete" wrt. the class of objects and the class of predicates: For each object or system of objects, resp., and for each predicate it is determined whether the predicate applies or not.

The notion of "possible world" has been introduced by Leibniz; in modern logic it is used as a model theoretic term. Possible worlds are constituents of model structures used to interpret a language, in particular modal concepts — necessity and possibility.

## Intensions

Def.: The *intension* of an expression  $A$  is the function which assigns to every possible world the extension of  $A$  in that world.

I.e., the intension of an expression determines its extensions in all possible worlds, and vice versa. Two expressions have the same intension iff they have the same extension in any possible world.

More precisely:

Assumption: For a set of possible worlds  $\mathcal{I}$  there is one and the same unique domain of individuals  $\mathcal{U}$ .

The intension of a nominator  $a$  is the function which determines for each possible world  $i \in \mathcal{I}$  the object which  $a$  denotes.

We interpret proper names as *standard names*, i.e. they denote the same object in all possible worlds, as opposed to pseudodescriptions (“Kennzeichnungsterme”):

$\iota_x p(x)$  “the  $x$  such that  $p(x)$ ”.

Example:

“Horst Koehler” vs. “The president of the Federal Republic of Germany”

The intension of a predicator  $p$  is the function which assigns for each possible world  $i \in \mathcal{I}$  the extension of  $p$  in  $i$ .

The intension of a sentence  $A$  is the function which assigns for each possible world  $i \in \mathcal{I}$  the truth value of  $A$  in  $i$ .

## Modality

A logical mode of truth- and falsehood of propositions — as opposed to the logical mode of inference.

Modalities are understood as operators which transform propositions or formulas into propositions or formulae, resp. (i.e., modifiers of truth values, where truth values are modalities, too).

*Motivation:* Sentences about the future, i.e. predictions.

Given a body of knowledge  $K$  consisting of a system of physical laws (of motion; development) and situation descriptions, we ask whether a sentence  $A[t]$ ,  $t > 0$  can be logically inferred:  $\Delta_K A[t] \Leftrightarrow K \prec A[t]$

Such conclusions  $A[t]$  from hypotheses are called “**necessary**” (relative to  $K$ ). But: Speaking of necessity makes just sense if implications among such modal propositions exist which are *independent* of a particular knowledge  $K$ .

## Reasoning with Necessity

Consider the meta-logical statement (about logical implications):

$$(K \prec A) \wedge (K \prec A \rightarrow B) \rightarrow (K \prec B)$$

$$\begin{array}{l} ? \Delta_K A \wedge \Delta_K(A \rightarrow B) \\ \quad ? \\ \quad \Delta_K A \\ \Delta_K(A \rightarrow B) \end{array} \parallel \begin{array}{l} \Delta_K A \wedge \Delta_K(A \rightarrow B) \rightarrow \Delta_K B \\ \Delta_K B \\ !? \\ r? \end{array}$$

Instead of prime formulas we get to  $\Delta$  formulas.

The position

$$\Delta_K A \quad \parallel \quad \Delta_K(A \rightarrow B) \quad \parallel \quad \Delta_K B$$

can be won for any  $K$  (Gentzen's main theorem — transitivity of implication).

Necessary premisses allow to infer a necessary conclusion if and only if one can infer *logically* without  $\Delta$  the conclusion from the premisses (Aristotle).

That means that a metalogical dialog position

$$\Delta_K A_1, \dots, \Delta_K A_n \parallel \Delta_K B$$

can be won for any  $K$  iff

$$A_1, \dots, A_n \parallel B$$

can be won *formally*.

## Modal Logic

The observation that one can infer relative necessities from other relative necessities without referring to a particular knowledge  $K$  yields the basis of a **modal logic**.

The  $\Delta$ -formulas are used as prime formulas, and the logical development steps are augmented by the  $\Delta$  step

$$\frac{\Sigma(\Delta A_1 \dots \Delta A_n) \parallel \Delta B}{\begin{array}{c} A_1 \\ \dots \\ A_n \end{array} \parallel B}$$

If a position  $\Sigma \parallel B$  with modal formulas can be closed in this way, we say that the system  $\Sigma$  implies the formula  $B$  modal-logically:  $\Sigma \prec B$ .

**Possibility:**  $\nabla A \Leftrightarrow \neg \Delta \neg A$   
**Contingency:**  $X A \Leftrightarrow \nabla A \wedge \neg \Delta A$

## Constructive Modal Logic Example

A constructive modal logic implication:

$$\frac{\begin{array}{c} \nabla_x \neg \Delta \neg a(x) \\ \Delta \neg \nabla_x a(x) \\ \neg \Delta \neg a(y) \end{array} \parallel \begin{array}{c} \neg \Delta \neg \nabla_x a(x) \\ \perp \end{array}}{\begin{array}{c} \neg \nabla_x a(x) \\ a(y) \end{array} \parallel \begin{array}{c} \Delta \neg a(y) \\ \neg a(y) \\ \perp \\ \nabla_x a(x) \\ a(y) \end{array}}$$

Remark: The inverse is not a modal logic implication!

## Classical Modal Logic

The restriction to *stable* modal formulas leads to classical modal logic: all  $\Delta$  formulas are interpreted as weakly affirmative:  $\neg\neg\Delta A$ , analogously for strong adjunctions (existential quantifiers).

The classical modal calculus contains for positions

$$\Sigma(\Delta A_1 \dots \Delta A_n) \parallel \Upsilon(\Delta B_1 \dots \Delta B_m)$$

with logically composed systems of  $\Delta$  formulas on the rhs *the same development steps* as the classical quantificational calculus, plus  $m$   $\Delta$  steps to place a  $B_i$  ( $1 \leq i \leq n$ ) on the rhs (instead of just the single  $B$ ).

A classical, but not constructive, modal-logically valid lemma:

$$\nabla(a \vee b) \prec \nabla a \vee \nabla b$$

## Kinds of Modalities

- ontic (*necessary, possible, . . .*)
- deontic (*imperative, permitted, . . .*)
- epistemic (*provable, irrefutable, . . .*, also *know, believe (rationally), . . .*)
- mellontic (temporal, wrt. to becoming)

Dependencies between modalities: implication graphs

$n$ -place modalities: compossibility  $\Leftrightarrow \nabla(A_1 \wedge \dots \wedge A_n)$

Lewis' "strict implication"  $\Leftrightarrow \Delta(A \rightarrow B)$

## Axiomatization of Modal Logic

**Syntactic** characterization of modal calculi:

(Additional) modal axioms for the propositional case:

System  $T$  or  $M$ , resp.:

$(N_1)$	$\Delta A \rightarrow A$	"necessity"
$(N_2)$	$\Delta(A \rightarrow B) \rightarrow (\Delta A \rightarrow \Delta B)$	"distributivity"
$(RN)$	$A \vdash \Delta A$	

System  $S_4$ : add

$(N_4)$	$\Delta A \rightarrow \Delta\Delta A$	"positive introspection"
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System  $S_5$ : add

$(N_5)$	$\nabla A \rightarrow \Delta\nabla A$	"negative introspection"
	$\neg\Delta A \rightarrow \Delta\neg\Delta A$	

Classical quantificational modal logic requires the *Barcan*-formula as an axiom:

$$(N_3) \quad \bigwedge_x \Delta A(x) \rightarrow \Delta \bigwedge_x A(x)$$

For this formula to be true, we require the "domain cumulation" principle: E.g., in temporal modal logic, the whole domain of quantification  $D(t)$  at time  $t$  is included in all the domains  $D(t')$  for times  $t'$  later than  $t$ .

**Remark:** Constructive quantificational logic can be simulated in classical quantificational  $S_4$ .

## Possible Worlds — Kripke Semantics

What about the semantic adequacy of the (purely) syntactically defined modal calculi? In particular, what about inhomogeneous modal formulas (e.g.,  $\Delta A \rightarrow A$ ) and iterated modalities??

In the classical approach a type of model theoretical semantics is sought which permits to prove completeness theorems for a variety of logical systems different from first order quantificational logic. Modal operators are understood as quantifiers whose domain is the set of possible worlds (or a subset restricted by a certain relation)  $\Rightarrow$  Enabling multiple reference.

So, interpretation of modal logic in Kripke semantics is done in a set theoretic extensional language where the intensions of object language sentence forms are interpreted as *extensional* functions from a set of "possible worlds" to the powerset of the domain.

*Brief sketch of the interpretation of modal formulas in intensional semantics:*

Prerequisite: One of the axiomatizations, usually  $S_4$ , is taken for granted.

Representation of possibility by a binary relation  $R$  on  $\mathcal{I}$ , the **accessibility relation**:

$R$  determines for each world  $I \in \mathcal{I}$  a set of worlds  $S_i = \{j \in \mathcal{I} \mid iRj\}$ ,  $S_i \neq \emptyset$ , which are accessible from  $i$ .  
(Example: temporal reading).

Then, the sentences possible in  $I$  are those which are true in at least one world in  $S_i$ , and the necessary sentences in  $i$  are true in all worlds in  $S_i$ . (\*)

Possible properties of  $R$ :

reflexivity  $wRw$

symmetry  $w_1Rw_2 \longrightarrow w_2Rw_1$

transitivity  $w_1Rw_2 \wedge w_2Rw_3 \longrightarrow w_1Rw_3$

euclidean  $w_1Rw_2 \wedge w_1Rw_3 \longrightarrow w_2Rw_3 \vee w_3Rw_2$

seriality  $\bigwedge_{w_1} \bigvee_{w_2} w_1Rw_2$

Symmetry of  $R$  leads to  $\Delta A \rightarrow A$ ,

transitivity to  $A \rightarrow \Delta \nabla A$ .

modal logic	properties of $R$	axioms
$K$	none	$N_2$
$M$	reflexive	$N_1, N_2$
$S_4$	reflexive, transitive	$N_1, N_2, N_4$
$S_5$	reflexive, transitive, symmetric	$N_1, N_2, N_4, N_5$
weak $S_4$	transitive	$N_2, N_4$
weak $S_5$	transitive, euclidean	$N_2, N_4, N_5$

So, in modal propositional logic each model consists of a set of possible worlds, an accessibility relation and an assignment of truth values to every elementary proposition in every possible world.

An interpretation of the language of modal logic is defined as a quadruple  $\langle \mathcal{U}, \mathcal{I}, R, f \rangle$  in the canonical way, where  $f$  is now a binary function which takes an  $i$  as its second argument (besides q.l. formulas as in the quantificational case). Additional requirement for modal formulas: (\*)

*Caution:* Quantification into modal contexts!

It is possible to represent Kripke semantics in dialogue logic where "possible worlds" are represented by dialogue levels and the accessibility relation corresponds to an admissibility relation between dialogue levels.

Advantage: A (purely) formally concept of modal logic admissibility.

## Applications of Modal Logic

- Inferences on knowledge ( $S_4, S_5$ )
- Inferences on belief (weak  $S_4, S_5$ )
- . . . in multiagent systems the modal operators are used as two-place operators where the first argument denotes an (knowing, believing) agent  $a$ . ( $\Rightarrow$  agent-specific  $R_a$ )  
Notation:  $\Delta_a F$  or  $\Box_a F$  or  $[a]F$
- Program verification: “Dynamic Logic”  
 $a$  denotes a program,  $R_a$  how “worlds”, i.e. states of an abstract machine, are related to each other by the program.

## Temporal Logics based on Modality

*Motivation:* Temporal operators ‘always’ and ‘sometimes’, defined analogously to ‘necessary’ and ‘possible’.

Theorems of such logics:

$$\begin{aligned} \text{always } q &\rightarrow \text{sometimes } q \\ \text{always } q &\rightarrow q \\ q &\rightarrow \text{sometimes } q \\ \text{sometimes } q &\rightarrow \neg \text{always } \neg q \end{aligned}$$

Augmentation the modalities ‘always’ and ‘sometimes’ with other modalities such as **P** for past (“it has at some time been the case that. . .”) and **F** for future (“It will at some time be the case that. . .”).

The resulting logics are called **tense logics** — pioneered by A. Prior et al.

Example formulas:

it-rains	meaning	it is raining
<b>P</b> (it-rains)	meaning	it rained
<b>PP</b> (it-rains)	meaning	it had rained
<b>F</b> (it-rains)	meaning	it will rain
<b>FP</b> (it-rains)	meaning	it will have rained

Other modalities may be defined in terms of **P** and **F**:

$\mathbf{H}q \Leftrightarrow \neg \mathbf{P}\neg q$	meaning	“it has always been the case that $q$ ”
$\mathbf{G}q \Leftrightarrow \neg \mathbf{F}\neg q$	meaning	“it will always be the case that $q$ ”

The relationship between **H**, **G** and ‘always’ may be expressed as:

$$\text{always } q \Leftrightarrow \mathbf{H}q \wedge q \wedge \mathbf{G}q$$

## The Minimal Propositional Tense Logic $\mathcal{K}_t$

Axioms:

$p \rightarrow \mathbf{H}p$	“what is, has always going to be”
$p \rightarrow \mathbf{G}p$	“what is, will always have been”
$\mathbf{H}(p \rightarrow q) \rightarrow (\mathbf{H}p \rightarrow \mathbf{H}q)$	“whatever always follows from what always has been, always has been”
$\mathbf{G}(p \rightarrow q) \rightarrow (\mathbf{G}p \rightarrow \mathbf{G}q)$	“whatever always follows from what always will be, always will be”

Temporal inference rules:

$$\begin{aligned} \text{RH: } & p \vdash \mathbf{H}p \\ \text{RG: } & p \vdash \mathbf{G}p \end{aligned}$$

The logic  $\mathcal{K}_t$  regards time as consisting of a linear sequence of states:  
 $\longrightarrow$  past  $\longrightarrow$  now  $\longrightarrow$  future  $\longrightarrow$

## Extended Tense Logics

- Branching time: possible futures
- Binary temporal operators “since” and “until”  
 $Spq$ : “ $q$  has been true since a time when  $p$  was true”  
 $Upq$ : “ $q$  will be true until a time when  $p$  is true”
- Metric tense logics
- “next time” operator: time as a sequence of atomic times
- Transition from the propositional logic base to quantificational logic:  
 Problems with the domain of quantification — always relative to a time

Semantics: closely modelled on the model theory of modal logic

The *model structure*  $E$  provides *possible denotations* that could be given to expressions of any language.

*Model assignment* assigns certain of these possible denotations to the expressions of some language.

- Altering the denotations of one-place predicates:  
 Instead of assigning to  $'P'$  subsets of  $E$  we assign them functions:  
 $E \rightarrow \{\top, \perp\}$   
 i.e. we identify a subset  $X \subset E$  with its characteristic function.  
 Reason: easier to generalize.
- Possible denotations relative to a model set  $E$ :  
 Introduction of **types**.  
  
 Basic types:  $e$  “entities” (possible individuals)  
 $t$  truth values  
 The set  $T^0$  of classical types is the smallest set s.t.  $e \in T^0, t \in T^0$ , and  
 whenever  $\sigma, \tau \in T^0$  then the pair  $\langle \sigma, \tau \rangle \in T^0$ .

## Intensional Type Logic (Montague)

Goal: Semantics for natural language (sentences) — “NL as a formal language”

- Starting point: “Disambiguated” language of propositional logic  $\mathcal{A}^0$   
 — fully bracketed version of  $\mathcal{L}$  with one-place predicates  $'P(a)'$ , where  
 $a$  is an individual constant; no variables, no quantifiers.
- Standard semantic theory for  $\mathcal{A}^0$   
 truth functions: truth values<sup>n</sup>  $\rightarrow$  truth values
- $\langle E, f \rangle$  is a model of  $\mathcal{A}^0$   
 $E \neq \emptyset, f$ : assignment  
 $f('a') \in E \rightarrow 'a'$  (basic expressions of  $\mathcal{A}^0$ )  
 $f('P') \subset E \rightarrow 'P'$   
 $f('¬') \subset E \rightarrow '¬'$  etc. ( $'\wedge', 'v'$ )

$\langle \sigma, \tau \rangle$ : the type of functions from things of type  $\sigma$  to things of type  $\tau$

Examples for types:  $e, t, \langle e, t \rangle, \langle t, t \rangle$  truth function of  $'¬'$ ,  
 $\langle t, t \rangle$  truth functions of  $'\wedge', 'v'$

- The set of possible denotations of type  $\tau$  relative to a model structure  $E$  is defined recursively:  
 $E$  is the set of possible denotations of type  $e$ .  
 $\{\top, \perp\}$  is the set of possible denotations of type  $t$ .  
 Whenever  $X$  is a set of possible denotations of type  $\sigma$ ,  
 and  $Y$  is a set of possible denotations of type  $\tau$ ,  
 $Y^X$  (the set of functions of  $X$  to  $Y$ ) is the set of possible denotations  
 of type  $\sigma, \tau$ .
- Strictly speaking: A model of a language assigns values only to basic expressions of the the language.  
 Assignment of values to composite expressions by means of semantic rules (one for each syntactic rule)  
 Extend  $f$  to  $f'$  which is defined for all expressions of  $\mathcal{A}^0$ :

- Basic expressions  $\zeta$ :  $f'(\zeta) = f(\zeta)$
- One-place predicate  $P$ , individual constant  $a$ :  
 $f'(F_0^0(P, a)) = (f'(P))(f'(a))$
- Formula  $\phi$ , one-place connective  $\zeta$ :  $f'(F_0^1(\zeta, \phi)) = (f'(\zeta))(f'(\phi))$
- Formulas  $\phi, \psi$ , two-place connective  $\zeta$ :  
 $f'(F_0^2(\zeta, \phi, \psi)) = (f'(\zeta))(f'(\phi), f'(\psi))$