

A CONSTRUCTIVE INTRODUCTION TO FIRST ORDER LOGIC

Goals of foundational programmes for logic:

- Supply an operational semantic basis for extant logic calculi (ex post)
- Rational reconstruction of the practice of argumentation
- Logic as a normative theory of argumentation

Approach:

Logic as a theory of (rational) argumentation — practice of arguing about claims of validity of sentences: *asserting, doubting, attacking, giving reasons*. . .

6. Davis, R.: *Logic, Deduction, and Computation*. New York: Computer Science Press, 1989
7. Schöning, U.: *Logik für Informatiker*. Mannheim: BI (HTB), 1992

Remark: Most texts on Dialogue Logic are in German!

References

1. Kamlah, W., Lorenzen, P.: *Logical Propaedeutic*. Lanham: Univ. Press of America, 1984
2. Lorenzen, P.: *Normative Logic and Ethics*. Mannheim: BI (HTB 236), 1969
3. Lorenzen, P.: *Constructive Philosophy*. Amherst: Univ. of Massachusetts Press, 1987
4. Lorenzen, P., Lorenz, K.: *Dialogische Logik*. Darmstadt: Wissenschaftliche Buchgesellschaft, 1978
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The Starting Point

Critical consideration of argumentation from its very beginning.

Idealized dialogue situations:

- No expectations, implicit assumptions, . . .
- All arguments required for the justification of a complex sentence are available (\Rightarrow "logically composed sentences");

Starting point: Begin with the simplest sentences — one word sentences in a **teaching / learning situation**: "Come!" "Bread?" "Small."

Predicators are *assigned* or *disassigned* to things: Examples and counterexamples

Proper names

Logical Propaedeutic

Elementary sentences

" E is p " $E\epsilon p$ or $p(E)$, resp.

" E is not p " $E\epsilon' p$

general form: $E_1, \dots, E_n \epsilon p$ etc.

"Logical Atomism": All sentences (of science) shall be reconstructed as logical compositions of atomic sentences.

Predicator rules (terminological rules): Stabilizing the use of predicators through the introduction of norms (imperatives) which apply to several predicators at the same time.

"Transition from ' $x\epsilon p$ ' to ' $x\epsilon q$ '!" $x\epsilon p \Rightarrow x\epsilon q$ (p : human, q : animal)

"Transition from ' $x\epsilon p$ ' to ' $x\epsilon' q$ '!" $x\epsilon p \Rightarrow x\epsilon' q$ (p : human, q : horse)

Rules for Argumentation

Set up a dialogue situation to conduct a normalized argumentation

- Two partners introduce *arguments* in turn,
- the form of which is determined by certain *particle rules*, and
- the admissibility of which is determined by certain *framework rules*.
- The dialogue is being conducted as a two-person (winning/losing) game about a (logically composed) proposition, the *thesis*.

Pragmatic foundation: Logical particles are introduced by means of rules to **perform** standardized (ideal) language games — language as action.

Logical Conjunction

Initial situation: Two (affirmative) predicator rules

$x\epsilon p \Rightarrow x\epsilon q_1, x\epsilon p \Rightarrow x\epsilon q_2$

Combination: $x\epsilon p \Rightarrow x\epsilon q_1 \wedge x\epsilon q_2$

— does not yet yield a justification

A norm has to be introduced on how propositions with form $A \wedge B$ are to be defended if they are denied.

Dialogue Game and Game Situation — Logical Conjunction

Opponent \mathcal{O} : denies the pretended proposition $A \wedge B$, doubts, "attacks" it
Proponent \mathcal{P} : defends the proposition

Attack: \mathcal{O} requires from \mathcal{P} , who pretended $A \wedge B$, that \mathcal{P} pretends one of the proposition's parts as chosen by \mathcal{O} (and defends it).

After \mathcal{P} has done that, in the new situation the ' \wedge ' has disappeared. If the proposition's parts have been defended, the conjunction has been defended.

Two Attack / defense rules

statement	attack	defense
$A \wedge B$	$L ?$	A
$A \wedge B$	$R ?$	B

By means of the sign ' \wedge ', the use of which has been standardized (normed) with these rules, two rules of the form $C \Rightarrow A, C \Rightarrow B$ can be combined into one: $C \Rightarrow A \wedge B$

Dialogue Game (2): Logical Adjunction

Analogical case:
 $A \Rightarrow C, B \Rightarrow C$ are admissible.
 Combination: $A \vee B \Rightarrow C$

Attack / defense rules

statement	attack	defense
$A \vee B$?	A
$A \vee B$?	B

\mathcal{O} is allowed to attack with '?' only, \mathcal{P} has the choice between A and B

Dialogue Game (3): Logical Subjunction

If systems of rules are accepted as admissible, we can justify "if . . . then . . ."

If $A \Rightarrow B, B \Rightarrow C$ are admissible, then the rule $A \Rightarrow C$ is admissible.

Proof: If \mathcal{O} has not denied A , he may not deny B according to the first rule, and furthermore he may not deny C according to the second rule.

Attack / defense rule

statement	attack	defense
$A \rightarrow B$	$A ?$	B

\mathcal{O} attacks $A \rightarrow B$ by asserting A .

\mathcal{P} can attack A (a counterattack), or defend himself by asserting B .

Dialogue Game (cont.)

Subjunction ' \rightarrow ' enables for each admissible rule $A \Rightarrow B$, to assert the composite statement $A \rightarrow B$ as true and to defend it:

Opponent	Proponent
1.	$A \rightarrow B$ (thesis)
2. $A ?$? 2 (counterattack)

Continuation: Either \mathcal{O} cannot defend A (loss), or he defends A . Then he is not allowed to deny B , because $A \Rightarrow B$ is admissible.

So \mathcal{P} wins by asserting B against the attack.

Dialogue Game (4): Logical Negation

Attack / defense rule

statement	attack	defense
$\neg A$	$A ?$	

The is no defense move; \mathcal{O} is only allowed to attack A .

Dialogue Game (5): Logical Quantifiers

Reconstruction of the use of “for all”, “for some”

Universal Quantifier: “All p are q ” \mapsto “For all $x: x \in p \rightarrow x \in q$ ”

Attack: \mathcal{O} chooses an arbitrary element E from the *variability domain* of x (e.g. proper names) such that \mathcal{P} is obliged to defend “ $E \in p \rightarrow E \in q$ ”.

Attack / defense rule

statement	attack	defense
$\bigwedge_x A(x)$	$E ?$	$A(E)$

Existential Quantifier: “Some p are q ”

Attack / defense rule (analogous)

statement	attack	defense
$\bigvee_x A(x)$	$?$	$A(E)$

Particle Rules: Overview

Proposition	Attack	Defense
$A \wedge B$	$l? \quad r?$	$A \quad B$
$A \vee B$	$?$	$A \quad B$
$A \rightarrow B$	$A ?$	B
$\neg A$	$A ?$	
$\bigwedge_x A(x)$	$n ?$	$A(n)$
$\bigvee_x A(x)$	$?$	$A(n)$

Assumption:

Propositions are **dialogue definite** (more general than **truth definiteness**).

Reasons: Application of logic to infinity (arithmetic) or to tense (future).

Dialogue Structure Rule

Considering logic as a theory of logical particles we have to stipulate how dialogues about (multiply) composite propositions are to be conducted: Who is allowed to perform a move at what time, and what are admissible attack and defense moves?

- *Opening rule*: \mathcal{P} begin with asserting a thesis.
- *Playing rule*: \mathcal{O} and \mathcal{P} deliver an argument (attack or defense) in turn. Attacks are rights, defenses are duties.
- *Winning rule*: The party which cannot submit an argument, has lost.

General (Effective) Dialogue Rule

“Last Duty First” — Rule of order:

Defense duties must to be delivered in the inverse order of their generation; i.e. the last one generated first.

Hence:

\mathcal{P} is allowed to attack a previous move of \mathcal{O} , or defend himself against \mathcal{O} 's last attack.

\mathcal{O} is allowed to attack *the last* move of \mathcal{P} , or defend himself against \mathcal{P} 's last attack.

\mathcal{P} has won if he has defended an attacked prime formula, or if \mathcal{O} cannot defend an attacked prime formula.

Consequences of the Dialogue Setting

(according to Inhetveen)

- Each dialogue turn (pair of attack and defense move) *eliminates* a junctor (the topmost one wrt. bracketing). The dialogue ends after a *finite* number of steps with elementary propositions — which have to be decided on “externally”.
- Dialogical meaning of subjunction: The defense may be delayed; hence an attack on the antecedent may be answered by a counterattack.
- In the case of subjunction or negation the opponent enters a situation in which he establishes a (partial) assertion, which then in turn can be attacked by the proponent: *Exchange of roles*.

Example Dialogue Game: Effective Dialogue Rule

	\mathcal{O}		\mathcal{P}
(1)			$\neg(a \wedge \neg b)$
(2)	$(a \wedge \neg b) ?$		$l ?_2$
(3)	a		$?_3$
(4)	$[a]$		$r ?_2$
(5)	$\neg b$		$b ?_5$
(6)	$?$		$[b]$

Remark: \mathcal{P} wins even then, if he can defend b only after \mathcal{O} has defended the proposition a (which is relevant only in the case when elementary propositions are not required to be truth definite).

Dialogue Games: Effective Dialogue Rule (2)

	\mathcal{O}	\mathcal{P}
(1)		$\neg\neg\bigwedge_x a(x) \rightarrow \bigwedge_x \neg\neg a(x)$
(2)	$\neg\neg\bigwedge_x a(x)$	$\bigwedge_x \neg\neg a(x)$
(3)	? n	$\neg\neg a(n)$
(4)	$\neg a(n)$	$\neg\bigwedge_x a(x)$
(5)	$\bigwedge_x a(x)$? n
(6)	$a(n)$	$a(n)$

(2): \mathcal{O} answers according to the rule for \rightarrow with the antecedent and \mathcal{P} subsequently with the consequent.

(3): \mathcal{O} asks for \mathcal{P} (2), \mathcal{P} answers.

(4): \mathcal{O} attacks \mathcal{P} (3), \mathcal{P} answers with attack on \mathcal{O} (2).

(5): \mathcal{O} attacks \mathcal{P} (4), \mathcal{P} asks for \mathcal{O} (5).

(6): \mathcal{O} answers, \mathcal{P} can attack \mathcal{O} (4) and wins.

So: \mathcal{P} is only obliged to defend the statement he pretended last, but \mathcal{O} must defend all statements pretended up so far.

Remark: Classical Dialogue Rule

\mathcal{P} is allowed to repeat attacks *and* defenses

$\hat{=}$ effective dialogue rule + stability hypotheses

(for each hypothesis A of \mathcal{P} add $\neg\neg A \rightarrow A$ on the lhs).

Example (2) Inverted

	\mathcal{O}	\mathcal{P}
(1)		$\bigwedge_x \neg\neg a(x) \rightarrow \neg\neg\bigwedge_x a(x)$
(2)	$\bigwedge_x \neg\neg a(x)$? n
(3)	$\neg\neg a(n)$	$\neg\neg\bigwedge_x a(x)$
(4)	$a(n)$	$\bigwedge_x a(x)$
(5)	? m	$a(m)$
	?	?

(2): \mathcal{P} has the choice: In the left branch he asks for \mathcal{O} (2), in the right branch he answers with the consequent.

(3.1): \mathcal{O} answers, \mathcal{P} attacks \mathcal{O} (3.1) — and loses if \mathcal{O} can reply in (4) with $a(n)$!

(3.2): \mathcal{O} attacks assertion \mathcal{P} (2.2), \mathcal{P} is forced to reply by attacking \mathcal{O}

(3.2) — and loses, if he cannot prove $a(m)$ for the particular m chosen by \mathcal{O} !

Hint:

\mathcal{P} would win with certainty only,

if he either knows an n such that \mathcal{O} cannot prove $a(n)$,

or if he himself can prove $a(m)$ for each m .

But he can have this knowledge only because of the contents of $a(x)$:

He cannot win the assertion because of its form alone.

Winning Strategies

Wanted: Winning Strategies for closed formulae, i.e.

Logical truth of a propositional form means to guarantee the possibility to win asserting any statement of this form.

We stipulate:

- (1) If \mathcal{O} pretends a closed prime formula, then \mathcal{P} is not allowed to deny it (it could be provable by \mathcal{O} in the special case).
- (2) \mathcal{P} has won, if he has to defend a closed prime formula which \mathcal{O} had already asserted (for any other prime formula it is possible that \mathcal{P} does not know the proof in that special case).

If the proponent has a winning strategy for the hypothesis dialogue on an initial position $(A_1, \dots, A_n) \parallel B$, we speak of a *logical inference* (*implication*) from a series of premisses A_1, \dots, A_n to a conclusion B .

Implications are relations on proposition schemata (sentence forms). The implication sign \prec is not a new particle, but a metalogical sign used for speaking about proposition schemata (and NOT to combine proposition schemata).

Theorem:

An implication $A_1, A_2, \dots, A_n \prec B$ is valid **iff** the proposition form $A_1 \wedge A_2 \dots \wedge A_n \rightarrow B$ is universally valid.

Winning Strategies (2)

A proposition is true, if for the proponent there is a way to conduct a dialogue s.t. he wins against any opponent: a *winning strategy*. If there is a winning strategy for the opponent the proposition is false.

A proposition is formally true if there is a winning strategy for the proponent s.t. he can take over any prime formula he has to defend from the opponent's side: formal winning strategy.

Rule for formal dialogues:

In a dialogue about the formal truth of a proposition,

- the opponent's prime formulae must not be attacked,
- the proponent may assert only such prime formulae which the opponent has asserted before.

Dialogue Trees

A winning strategy can be represented as a **dialogue tree**:

- it contains all possible choices of the opponent,
- it contains one resp. possible choice of the proponent,
- all branches of the dialogue tree are won by the proponent.

What are the rules for winning strategies?

FORMAL DIALOGUES AND TABLEAUX

Formal logic:

Transition from "material" dialogues to formal developments (tableaux).

Rules for winning strategy:

Rules of the sequent calculus (Gentzen's G3) read in the inverse direction!

⇒ Development rules for winning strategies (⇔ cf. particle rules):

Let Σ denote all formulas set by the opponent so far; $\Sigma[A]$ means that formula A occurs in Σ .

$$\Sigma \left\| \begin{array}{l} A \wedge B \\ A \mid B \end{array} \right\| C \qquad \Sigma[A \wedge B] \left\| \begin{array}{l} C \\ A \end{array} \right\| B \qquad \Sigma[A \wedge B] \left\| \begin{array}{l} C \\ B \end{array} \right\| C$$

$$\Sigma \left\| \begin{array}{l} \bigwedge_x A(x) \\ A(n) \text{ f.a. } n \end{array} \right\| C \qquad \Sigma[\bigwedge_x A(x)] \left\| \begin{array}{l} C \\ A(n) \end{array} \right\| C$$

$$\Sigma \left\| \begin{array}{l} A \vee B \\ A \end{array} \right\| \Sigma \left\| \begin{array}{l} A \vee B \\ B \end{array} \right\| \Sigma[A \vee B] \left\| \begin{array}{l} C \\ A \mid B \end{array} \right\| C$$

$$\Sigma \left\| \begin{array}{l} \bigvee_x A(x) \\ A(n) \end{array} \right\| \Sigma[\bigvee_x A(x)] \left\| \begin{array}{l} C \\ A(n) \end{array} \right\| \text{f.a. } n$$

$$\Sigma \left\| \begin{array}{l} \neg A \\ A \end{array} \right\| \perp \qquad \Sigma[\neg A] \left\| \begin{array}{l} C \\ A \end{array} \right\| C$$

$$\Sigma \left\| \begin{array}{l} A \rightarrow B \\ A \end{array} \right\| B \qquad \Sigma[A \rightarrow B] \left\| \begin{array}{l} C \\ \mid B \end{array} \right\| A \mid$$

A branch which contains the same prime formula a on the left **and** on the right is called (formally) **closed**.

⇒ Formal tableaux (Beth): Logical truth ⇔ formal closure

Example: Winning Strategy for MP

Formal winning strategy for the logical inference (inference schema)

$$A, A \rightarrow B \prec B \quad (\text{modus ponens})$$

Dialogue with hypotheses:

(0)	A	
(1)	A → B	B
(2)	?	...
(3)	B	[A] ? ₁
(4)		[B] 2

Remark: In the development calculus line (2) is left out.

Notabene: The theorem $\neg(a \wedge \neg a)$ can be (formally) closed constructively, but **not** the tertium-non-datur $a \vee \neg a$!

Expressiveness of the Dialogue Framework

Constructive (effective, dialogue definite) as well as "classical" (two valued truth definite) logic calculi can be handled.

"Classically valid" is equivalent to "irrefutable" (without knowing the particular justification or refutation).

Reasons for constructive (effective) logic (Thiel)

- avoiding the restriction to decidable domains,
- finding inference rules which never lead to existential formulae or adjuncts as conclusions where these are not effectively contained in the premisses (no proofs by contradiction without explicit construction).
- Comparison with Kolmogorov's interpretation as "problem logic" (Aufgabenlogik).
- Difference to "multivalued logics": The lack of knowledge of a truth value is not a third truth value!

Fundamental vs. Interpretation Semantics

- Traditionally: The concept of logical deduction defined by derivation rules and axioms must be justified
 - The concept of inference is defined within an interpretation semantics;
 - To be shown: The concepts of deduction and inference are equivalent (correctness and completeness)
- Problem: Interpretation semantics cannot solve the foundation problem of logic, because it presupposes on the metalevel in advance what has to be justified — as well for the particles as for the inference relation (circularity).
- Solution: Replace the name theory of meaning by the theory of use — The meaning of the junctors and quantifiers is its use in the language (explication of the rules of use).

- One possibility: *Dialogue* \Rightarrow **Fundamental Semantics**
- The parts of an interpretation semantics relevant for metalogical investigations can be reconstructed as a structure theory (Hinst).

Formal Dialogues and Tableaux

Formal dialogue logic:

Transition from “material” dialogues to formal developments (tableaux).

Rules for winning strategy:

Rules of the sequent calculus (Gentzen’s G3) read in the inverse direction!
(\Leftrightarrow cf. particle rules)

\Rightarrow Formal tableaux (Beth): Logical truth \Leftrightarrow formal closure

Handling in the dialogue framework:
constructive [effective, dialogue definite) and
“classical” (two valued truth definite) logic calculi.

“Classically valid” is equivalent to “irrefutable” (without knowing the particular justification or refutation).

Formal Dialogues and Tableaux (2)

Connection between the classical introduction of formal logic (Hilbert), Gentzen’s sequent calculi of “natural deduction”, and Beth’s tableaux calculi:

Hilbert: Formal derivation of (classical) logically true formulas as theorems of an axiomatic calculus.

Inference rule: modus ponens $A, A \rightarrow B \vdash B$

Gentzen’s sequent calculi of “natural reasoning”: Derivation of logical implications — instead of logically true formulas —, written as sequences. Specific sequences are axioms; sequences are derived from sequences by means of inference rules.

Particular property: The premisses of the rules consist of subformulas of the formulas in their conclusion (with the exception of the cut rule).

Beth’s tableau calculus: applies sequent calculus in inverse direction. Each development begins with a sequence (thesis) and applies inverted sequent rules until prime formulas are obtained in all branches (\Leftrightarrow form. dialogues)

Tableau Calculus

Tableau Calculus is a decision procedure to solve the satisfiability problem.

If a formula is satisfiable, the procedure leads constructively to a model of the formula.

The basic idea is to construct the model in a stepwise fashion by analyzing the formula and decomposing it “top down”. The procedure considers all possibilities exhaustively such that it can eventually prove that no model can be found for unsatisfiable formulas.